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Phenomenological mapping and dynamical absorptions in chain systems with multiple degrees of freedom

Katica R Stevanović Hedrih[1,2] and Andjelka N Hedrih[3]

Abstract

Using Mihailo Petrović's theory of mathematical phenomenology elements, phenomenological mapping in vibrations, signals, resonance and dynamical absorptions in models of dynamics of chain systems – the abstractions of different real dynamics of a chain system are identified and presented. Using a mathematical description of a chain mechanical system with a finite number of mass particles coupled by linear elastic springs and a finite number of degrees of freedom expressed by corresponding generalized independent coordinates, translator displacements and corresponding analysis of solutions for a free and forced vibrations series of multi-frequency regimes and resonant states as well as dynamical absorption states are identified. Using mathematical analogy and phenomenological mapping, analyses of the dynamics of other chain models are made. Phenomenological mapping is used to explain dynamics in systems with multiple deformable bodies (beams, plates, membranes or belts) through resonance and dynamical absorptions in the system and transfer of mechanical energies between bodies. Amplitude-frequency graphs for homogeneous and non-homogeneous chain systems are presented for a system with 11 degrees of freedom. Expressions for generalized coordinates of a chain non-homogeneous system in resonance regimes for a general case are derived. A theorem is defined and proven.

Keywords

Chain systems, discrete continuum method, multi-body systems, phenomenological mapping, vibrations

1. Introduction

In two books Mihailo Petrović Alas presented a theory containing elements of mathematical phenomenology and phenomenological mapping (Petrović, 1911, 1933). Both publications were published in Serbian, and only a small number of his contemporaries were able to read and understand this theory. Alas's theory defines two types of analogy: qualitative and mathematical analogy. In the time of expansion of computers and software tools, Roger Penrose (1989) and James Gleick (1987) had similar ideas that were later applied in graphical-computer techniques.

Phenomenological mapping of phenomenons and models enables multiple dynamics of system models of disparate natures to be described by a single mathematical model: for example an electric chain model and a model of a mechanical chain with the same degree of dynamics freedom.

Both the mechanical signal through a chain and an electrical signal can be described with the same equations, although they describe different physical phenomena. In both systems a set of circular frequencies, resonance, dynamical absorption and signal filtering are in question regardless of whether it is mechanical motion or an electrical signal.

This approach makes possible an integration of scientific knowledge and a reduction of the qualitative models or corresponding mathematical models that are needed. Hence, it is possible to describe the dynamics of real systems with various physical properties,

[1]Department for Mechanics, Mathematical Institute SANU Belgrade, Serbia
[2]Faculty of Mechanical Engineering, University of Niš, Serbia
[3]Department for Bio-medical Science, State University of Novi Pazar, Serbia

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Corresponding author:
Andjelka N Hedrih, Department for Bio-medical Science, State University of Novi Pazar, Vuka Karadzica bb, 36 300 Novi Pazar, Serbia.
Email: handjelka@hm.co.rs

using phenomenological mapping to map the phenomena from one system to another.

On the basis of this theory it is possible to integrate contemporary knowledge obtained in various areas of sciences and identify analogous dynamics and phenomena.

As far we know, there are few studies available in the literature discussing phenomenological mapping: in mathematics (Frazer, 1961; Freudenthal, 1986; Pettifor, 1986), from the area of neuroscience (Walsh, 1995) or as a part of string theory and its applications (Kelebanov and Maldacena, 2009). String theory is a self-contained mathematical model that describes all fundamental forces and forms of matter.

In classical books, the dynamics of chain systems are usually described as systems for up to three degrees of freedom (d.f.).

Pfeiffer et al. (1997) gave examples of chain dynamics in practical engineering problems (machine components in cars, in gears, and in armored vehicles). The dynamics of chains are typically characterized by free motion and contact processes, which may include impacts and friction. Therefore, modeling by multibody theory augmented by methods of contact mechanics represents an appropriate way to evaluate dynamic chain behavior. Different types of resonance in linear and nonlinear system forced dynamics are very important phenomena in linear and nonlinear system forced dynamics. The results of investigations of primary and secondary resonances of the first mode in MEMS electrostatic microactuators are presented by Najar et al. (2010). These authors use a discretization technique that combines the differential quadrature method and the finite difference method for space and time, respectively, to study the dynamic behavior of a microbeam-based electrostatic microactuator. The method is applied for large excitation amplitudes and large quality factors for primary and secondary resonances of the first mode in the case of hardening-type and softening-type behaviors. The occurrence of dynamic pull-in due to subharmonic and superharmonic resonances is identified. Also, it is identified that the excitation amplitude is increased. Simultaneous resonances of the first and higher modes are identified for large orbits in both primary and secondary resonances. Filipovic and Schroder (1999) gave a concept of the linear active resonator as a vibration absorber. It is formed of a classical passive absorber with a simple dynamic linear feedback. These authors investigated vibration absorption with linear active resonators, containing continuous and discrete time design and analysis. Ekwaro-Osire and Desen (2001) investigated the effects of mass ratio, clearance, and excitation amplitude on the dynamics of a system and the effectiveness of impact vibration absorbers. The experimental studies were carried out for both free and forced vibrations. For free vibrations, the effects of system parameters on the rate of decay of vibrations were shown.

Models of resonance and dynamical absorption in chain system forced dynamics in pure linear systems with more then three d.f. are missing in the world literature. As far as we know there are no results for qualitative analogy and mathematical analogy between the dynamics of chain systems with more than three d.f.. Investigating resonance and dynamical absorption in dynamics in chain systems with more than three d.f. using phenomenological mapping even in the area of classical and linear chain forced dynamics is important not only for mechanical signal processing, but also for electrical signal processing and signal filtering, for processing biodynamical signals in life systems (DNA double chain helix (Hedrih (Stevanović) and Hedrih, 2010), biodynamical chain oscillators (Hedrih, 2011, 2012) and also for teaching in university and for integrations of scientific results in different areas of science.

For all the reasons mentioned above the aim of our research was to investigate classical models of free and forced dynamics in linear chains with a finite number of d.f., and with corresponding numerical analysis of a system with 11 d.f.; to analyze the existence of dynamical absorption at an amplitude of mass particles' displacement in a forced regime of the system's vibrations.

The phenomenon defined as *dynamical absorption* appears only in conservative systems with two and more d.f. in forced regimes when a single frequency force is applied at one mass particle, and this mass particle is at forced rest, and other mass particles in the system vibrate in forced regimes. As far as we know, this phenomenon is not described in enough detail or investigated in the literature for systems with more than three d.f., as well as numerous resonances. This forced vibration phenomena is taken into considertion. Also, we derived expressions for solutions for mass particle resonant displacements.

Elements of mathematical phenomenology – qualitative and mathematical analogy – are used to make a transfer for all results obtained for forced dynamics of a mechanical chain system to other chain systems, such as a torsion chain system (gear chain machines), a multipendulum chain system, an electrical circle chain system's forced vibrations, and the dynamics of DNA helix (Hedrih (Stevanović) and Hedrih, 2010). Also, chain dynamics is analagous qualitatively and mathematically with the forced vibrations of an elastically coupled multi-deformable body system (coupled beams, coupled plates, coupled belts, coupled membranes, for the same boundary conditions) (Hedrih (Stevanović), 2006a,b, 2008a,b).

Rašković (1952, 1972, 1974) gave a series of examples of electromechanical mathematically analogous vibration systems that were mathematically described and solved for free vibrations. Hedrih (Stevanović) (1991) presents the analogy between vector models of stress state, strain state and state of the body mass inertia moments. Ideas of mathematical phenomenology and phenomenological mappings, from listed references, are used for investigating the dynamics and vibration phenomena of resonance and dynamical absorption for solving a series of research problems of dynamics of various kinds of chain systems.

Four mathematically analogous chain systems with qualitative analogous vibration phenomena (dynamical absorptions, resonance, and sets of eigen circular frequencies) are presented in Figure 1. In Figure 1(a) a chain mechanical system with a finite number of mass particles with masses m_k, coupled by springs with rigidity c_k and a finite number of d.f. expressed by corresponding generalized independent coordinates – translator displacements $x_k,\ k = 1, 2, \dots, N$ is presented; in Figure 1(b) a chain mechanical torsion system with a finite number of disks with mass inertia axial moments J_k coupled by a shaft with different torsion rigidity c_k and a finite number of d.f. expressed by generalized angular coordinates $\varphi_k,\ k = 1, 2, \dots, N$ is presented; in Figure 1(c) a chain mechanical multi-pendulum system with a finite number of mass particles with masses m_k, on the same length ℓ coupled with spring rigidity c_k and a finite number of d.f. expressed by generalized angular coordinates $\varphi_k, k = 1, 2, \dots, N$ is presented; and in Figure 1(d) a chain electrical circuit system with a finite number of coil inductances L_k coupled by capacitor capacitances C_k and a finite number of d.f. expressed by generalized coordinates – electrical charge of capacitor $q_k, k = 1, 2, \dots, N$ or velocity of generalized coordinate – $i_k = \frac{dq_k}{dt}, k = 1, 2, \dots, N$ intensity of electric current flowing through the branch circuits is presented. The dynamics of previously listed chain system models presented in Figure 1 may be described by the same type of system of ordinary differential equations. Between systems, we identify elements of mathematical analogy and the following elements of mathematical phenomenology. It is enough to write a system of ordinary differential equations for free and forced dynamics of a chain mechanical system presented in Figure 1(a), find corresponding solutions with graphical presentations, and after identifying characteristic phenomena, by phenomenological mapping it is possible to make analysis of the dynamics of all the other three systems.

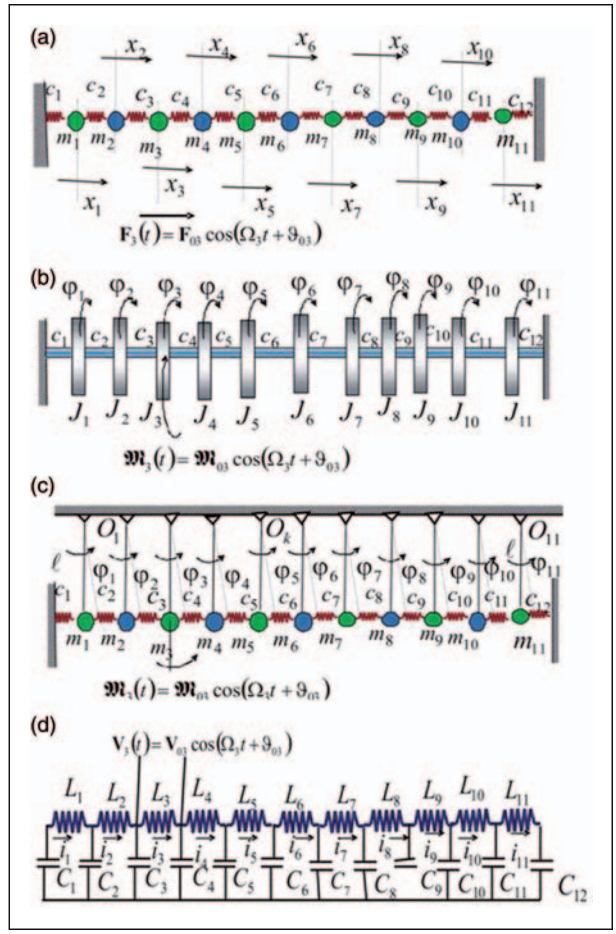

Figure 1. Analogous chain systems: (a) Chain mechanical system with finite number of mass particles with masses m_k, coupled by springs with rigidity c_k and finite number of degrees of freedom expressed by corresponding generalized independent coordinates – translator displacements x_k, $k = 1, 2, \dots, N$; (b) Chain mechanical torsion system with finite number of disks with mass inertia axial moments J_k coupled by shaft with different torsion rigidity c_k and finite number of degrees of freedom expressed by generalized angular coordinates φ_k, $k = 1, 2, \dots, N$; (c) Chain mechanical multi-pendulum system with finite number of mass particles with masses m_k, on the same length ℓ coupled with spring rigidity c_k and finite number of degrees of freedom expressed by generalized angular coordinates φ_k, $k = 1, 2, \dots, N$; (d) Chain electrical circuit system with finite number of coil inductances L_k coupled by capacitor capacitances C_k and finite number of degrees of freedom expressed by generalized coordinates – electrical charge of capacitor q_k, $k = 1, 2, \dots, N$ or velocity of generalized coordinate – $i_k = \frac{dq_k}{dt}$, $k = 1, 2, \dots, N$ intensity of electric current flowing through the branch circuits

2. Free vibrations of homogeneous chains

Homogeneous mechanical chains as well as electrical chains are used as filters and they may miss certain frequencies or frequency ranges (Rašković, 1952, 1974; Hedrih (Stevanović), 1991). Depending on the range of frequencies that may be missing there are low-frequency and high-frequency filters.

In Rašković (1952) a system of the ordinary differential equations of the dynamics of a homogeneous chain mechanical linear system is solved by using a trigonometric method for different cases of springs at both ends. Using these results for free vibrations of a chain with both ends fixed ($x_0 = 0$, $x_{N+1} = 0$) described by following system of ordinary differential equations

$$\frac{m}{c}\ddot{x}_k = -(x_k - x_{k-1}) + (x_{k+1} - x_k), \quad k = 1, 2, \dots, N \tag{1}$$

we can write the following:

- set of eigen circular frequencies of homogeneous chain system free oscillations in the form

$$\omega_s = 2\sqrt{\frac{c}{m}}\sin\frac{\varphi_s}{2} = 2\sqrt{\frac{c}{m}}\sin\frac{s\pi}{2(N+1)} \quad \text{where} \quad s = 1, 2, 3, \dots, N \tag{2}$$

and

- solutions of a system of ordinary differential equations (1) of a homogeneous chain system free oscillations in the form

$$x_k = \sum_{s=1}^{s=N}\xi_s(t)\sin\frac{ks\pi}{(N+1)} = \sum_{s=1}^{s=N}K_{Nk}^{(s)}\xi_s(t), \quad k = 1, 2, 3, \dots, N \tag{3}$$

which present mass particle longitudinal multi-frequency vibration displacements, and in which $\xi_s(t)$ are eigen main coordinates of system dynamics in the form

$$\xi_s(t) = D_s\cos(\omega_s t + \gamma_s), \quad s = 1, 2, 3, \dots, N \tag{4}$$

where D_s and γ_s are integral constants depending on initial conditions. By the use of cofactors $K_{Nk}^{(s)} = \sin\frac{ks\pi}{(N+1)}$ it is correct to form eigen main vectors and corresponding modal matrix in the form

$$\mathbf{R} = \left(K_{Nk}^{(s)}\right)_{\to\, s=1,2,3,\dots,N}^{\downarrow \quad k=1,2,3,\dots,N} = \left(\sin\frac{ks\pi}{(N+1)}\right)_{\to\, s=1,2,3,\dots,N}^{\downarrow \quad k=1,2,3,\dots,N} \tag{5}$$

and in matrix form express independent generalized coordinates $x_k(t)$ by eigen main coordinates $\xi_s(t)$

$$\left\{x_k(t)\right\}^{\downarrow \quad k=1,2,3,\dots,N} = \left(K_{Nk}^{(s)}\right)_{\to\, s=1,2,3,\dots,N}^{\downarrow \quad k=1,2,3,\dots,N}\left\{\xi_s(t)\right\}^{\downarrow\, s=1,2,3,\dots,N} \tag{6}$$

As it is for: $N \to \infty$ $\omega_1 = \lim_{N\to\infty} 2\sqrt{\frac{c}{m}}\sin\frac{\pi}{2(N+1)} = 0$ and $\lim_{N\to\infty}\omega_N = \lim_{N\to\infty} 2\sqrt{\frac{c}{m}}\sin\frac{N\pi}{2(N+1)} = 2\sqrt{\frac{c}{m}}$, then it can be concluded that for homogeneous chains with both ends fixed, a set of eigen circular frequencies is in the interval $\omega_s \in (0, 2\sqrt{\frac{c}{m}})$.

Kinetic and potential energies for free vibrations of the considered homogeneous chain system are

$$\begin{aligned} E_{kin} &= \frac{1}{2}\sum_{k=1}^{k=N} m_k\dot{x}_k^2 \\ &= \frac{1}{2}(\dot{x})\mathbf{A}\{\dot{x}\} = \frac{1}{2}(\dot{\xi}_s)\mathbf{R}'\mathbf{A}\mathbf{R}\{\dot{\xi}_s\} \\ &= \frac{1}{2}m(\dot{\xi}_s)\tilde{\mathbf{A}}\{\dot{\xi}_s\} = \frac{1}{2}(\dot{\varsigma}_s)\mathbf{I}\{\dot{\varsigma}_s\} \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{R}'\mathbf{A}\mathbf{R} \\ &= \left(\sin\frac{ks\pi}{(N+1)}\right)_{\to\, k=1,2,3,\dots,N}^{\downarrow \quad s=1,2,3,\dots,N}\mathbf{A}\left(\sin\frac{ks\pi}{(N+1)}\right)_{\to\, s=1,2,3,\dots,N}^{\downarrow \quad k=1,2,3,\dots,N} \\ &= diag(a_{ss}) \end{aligned} \tag{8}$$

The potential energy of the system is in the form

$$\begin{aligned} E_{pot} &= \frac{1}{2}\sum_{k=1}^{k=N+1} c_k(x_k - x_{k-1})^2 = \frac{1}{2}(x)C\{x\} \\ &= \frac{1}{2}(\xi_s)\mathbf{R}'C\mathbf{R}\{\xi_s\} = \frac{1}{2}m(\xi_s)\tilde{C}\{\xi_s\} \\ &= \frac{1}{2}(\varsigma_s)\Lambda\{\varsigma_s\} \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{\mathbf{C}} &= \mathbf{R}'\mathbf{C}\mathbf{R} \\ &= \left(\sin\frac{ks\pi}{(N+1)}\right)_{\to\, k=1,2,3,\dots,N}^{\downarrow \quad s=1,2,3,\dots,N}\bar{\mathbf{C}}\left(\sin\frac{ks\pi}{(N+1)}\right)_{\to\, s=1,2,3,\dots,N}^{\downarrow \quad k=1,2,3,\dots,N} \\ \tilde{\mathbf{C}} &= diag(c_{ss}) \end{aligned} \tag{10}$$

$$\Lambda = diag\left(\omega_s^2\right) = diag\left(\frac{c_{ss}}{a_{ss}}\right), \quad s = 1, 2, 3, \dots, N \tag{11}$$

The total mechanical energy of the homogeneous chain system for free vibration regimes is

$$\begin{aligned} E &= E_{kin} + E_{pot} \\ &= \frac{1}{2}\left[\sum_{k=1}^{k=N} m_k\dot{x}_k^2 + \sum_{k=1}^{k=N+1} c_k(x_k - x_{k-1})^2\right] = const \end{aligned}$$

$$E = \frac{1}{2}(\dot{x})\mathbf{A}\{\dot{x}\} + \frac{1}{2}(x)C\{x\}$$
$$= \frac{1}{2}m(\dot{\xi}_s)\tilde{\mathbf{A}}\{\dot{\xi}_s\} + \frac{1}{2}m(\xi_s)\tilde{C}\{\xi_s\} \quad (12)$$
$$= \frac{1}{2}(\dot{\varsigma}_s)\mathbf{I}\{\dot{\varsigma}_s\} + \frac{1}{2}(\varsigma_s)\Lambda\{\varsigma_s\} = const$$

Where $\mathbf{I} = diag(1 \quad 1 \quad \dots \quad 1)$ is unit diagonal matrices, $\Lambda = diag(\omega_1^2 \quad \omega_2^2 \quad \dots \quad \omega_{11}^2)$ is the matrix of the square of eigen circular frequences, $\xi_s(t) = D_s \cos(\omega_s t + \gamma_s)$ is the eigen main coordinate, and $\varsigma_s(t) = \tilde{D}_s \cos(\omega_s t + \gamma_s)$, is the eigen normal coordinate, D_s, $\tilde{D}_s$ and γ_s are integral constants determined by initial conditions. Total mechanical energy of the mode eigen circular frequency ω_s as a sum of kinetic and potential energy is in the form

$$E_s = \frac{1}{2}m\left\langle\dot{\xi}_s^2(t) + \omega_s^2\xi_s^2(t)\right\rangle$$
$$= \frac{1}{2}\left\langle\dot{\varsigma}_s^2(t) + \omega_s^2\varsigma_s^2(t)\right\rangle \quad (13)$$

$$E_s = 2c\,\sin^2\frac{s\pi}{2(N+1)}D_s^2 = constatnt \quad (14)$$

If the homogeneous chain system is nonlinear with nonlinear cubic nonlinearities of the ideal elastic springs between mass particles:

- Potential energy of the free dynamics of system is in the form

$$E_{pot} = \frac{1}{2}\sum_{k=1}^{k=N+1} c_k(x_k - x_{k-1})^2$$
$$+ \frac{1}{4}\sum_{k=1}^{k=N+1} \tilde{c}_k(x_k - x_{k-1})^4 \quad (15)$$

- The total mechanical energy of the dynamics of a nonlinear chain system for free vibrations is in the form

$$E = E_{kin} + E_{pot}$$
$$= \frac{1}{2}\sum_{k=1}^{k=N} m_k\dot{x}_k^2$$
$$+ \frac{1}{2}\sum_{k=1}^{k=N+1} c_k(x_k - x_{k-1})^2$$
$$+ \frac{1}{4}\sum_{k=1}^{k=N+1} \tilde{c}_k(x_k - x_{k-1})^4 = const \quad (16)$$

Using eigen main coordinates of corresponding linear generalized coordinates of a nonlinear chain system in the form (3) and taking into account that these eigen main coordinates $\xi_s(t)$ are unknown functions of time in the form $\xi_s(t) = D_s(t)\cos(\omega_s t + \phi_s(t))$ and in which amplitude $D_s(t)$ and phase $\phi_s(t)$ are functions of time defined by a system of ordinary differential equations in corresponding asymptotic approximation, it is possible to write an expression for total mechanical energy of the dynamics of the nonlinear chain system in the following form

$$E = E_{kin} + E_{pot}$$
$$= \frac{1}{2}m(\dot{\xi}_s)\tilde{\mathbf{A}}\{\dot{\xi}_s\} + \frac{1}{2}m(\xi_s)\tilde{C}\{\xi_s\}$$
$$+ \frac{1}{4}\sum_{k=1}^{k=N+1}\tilde{c}_k\left(\sum_{s=1}^{s=N}\left\langle K_{Nk}^{(s)} - K_{N(k-1)}^{(s)}\right\rangle\xi_s(t)\right)^4 = const \quad (17)$$

By analyzing the previous expression, we can make conclusions in the form of two well known theorems:

Theorem 1. Total mechanical energy of free nonlinear vibrations of a mechanical chain system with a finite number of mass particles coupled by nonlinear ideal elastic springs is constant for all vibrations and equal to the total mechanical energy of the system at the initial moment. There are transfers of energy between mass particles as well as from kinetic energy to potential energy and the opposite way round.

Theorem 2. Total mechanical energy carried by s-th nonlinear main mode in nonlinear chain mechanical dynamics of a system is not constant, and there are interactions and transfers of energy between each s-th and r-th nonlinear main modes. The sum of all total mechanical energies carried by all nonlinear main modes in the nonlinear chain mechanical dynamics of a system for free vibrations is constant, and equal to the total mechanical energy of the system at the initial moment.

3. Forced vibrations of homogeneous chains

Consider the forced vibrations of a homogeneous mechanical chain with 11 mass particles and 11 d.f. presented in Figure 1(a) and excited by external one frequency excitation $F_3(t) = F_{03}\cos(\Omega_3 t + \phi_{03})$, amplitude F_{03}, circular frequency Ω_3, and phase ϕ_{03}, applied to the third mass particle along a longitudinal chain direction. Taking denotation $h_3 = \frac{F_{03}}{c}$, a system of ordinary differential equations for forced longitudinal vibrations is in the following form

$$\begin{aligned}\frac{m}{c}\ddot{x}_k &= -(x_k - x_{k-1}) \\ &\quad + (x_{k+1} - x_k) + \delta_{3k} h_3 \cos(\Omega_3 t + \phi_{03}) \\ k &= 1,2,3,4,5,6,7,8,9,10,11\end{aligned} \tag{18}$$

where δ_{3k} is the Kronecker symbol.

Suppose that the particular solutions of the previous system (18) of ordinary differential equations is in the form

$$x_{(p)k} = C_k \cos(\Omega_3 t + \phi_{03}) \tag{19}$$

then taking denotation $v_3 = \frac{m}{c}\Omega_3^2$ it is possible to write the determinant of the obtained non-homogeneous algebra equations along unknown amplitudes C_k

$$\Delta(v_3) = \begin{vmatrix} 2-v_3 & -1 & & & & & & & & & \\ -1 & 2-v_3 & -1 & & & & & & & & \\ & -1 & 2-v_3 & -1 & & & & & & & \\ & & -1 & 2-v_3 & -1 & & & & & & \\ & & & -1 & 2-v_3 & -1 & & & & & \\ & & & & -1 & 2-v_3 & -1 & & & & \\ & & & & & -1 & 2-v_3 & -1 & & & \\ & & & & & & -1 & 2-v_3 & -1 & & \\ & & & & & & & -1 & 2-v_3 & -1 & \\ & & & & & & & & -1 & 2-v_3 & -1 \\ & & & & & & & & & -1 & 2-v_3 \end{vmatrix} \tag{20}$$

Corresponding sub-determinants $\Delta_k(v_3)$, $k = 1,2,3,4,\ldots,11$: it is possible to obtain by substituting the corresponding k-th column by column $\{\delta_{3k} h_3\}$ containing the non-zero element h_3 into the determinant of system $\Delta(v_3)$ expressed in (20). Where h_3 is the reduced amplitude of external excitation force applied to the third mass particle of the chain system. It can be written for the first subdeterminant $\Delta_1(v_3)$

$$\Delta_1(v_3) = -h_3 \begin{vmatrix} -1 & & & & & & & & & \\ 2-v_3 & -1 & & & & & & & & \\ & -1 & 2-v_3 & -1 & & & & & & \\ & & -1 & 2-v_3 & -1 & & & & & \\ & & & -1 & 2-v_3 & -1 & & & & \\ & & & & -1 & 2-v_3 & -1 & & & \\ & & & & & -1 & 2-v_3 & -1 & & \\ & & & & & & -1 & 2-v_3 & -1 & \\ & & & & & & & -1 & 2-v_3 & -1 \\ & & & & & & & & -1 & 2-v_3 \end{vmatrix} \tag{21}$$

On the basis of Cramer Low, unknown amplitudes of particular solutions are possible to write in the following form

$$C_{(3)k}(v_3) = \frac{\Delta_k(v_3)}{\Delta(v_3)}, \text{ or in the form } C_{(3)k}\left(\Omega_3^2\right) = \frac{\Delta_k\left(\Omega_3^2\right)}{\Delta\left(\Omega_3^2\right)}$$
$$\text{where} \quad v_3 = \frac{m}{c}\Omega_3^2 \tag{22}$$

And a particular solution is in the following form

$$x_{(p)k} = \frac{\Delta_k\left(\Omega_3^2\right)}{\Delta\left(\Omega_3^2\right)}\cos(\Omega_3 t + \phi_{03}) \tag{23}$$

General solutions of the system of ordinary differential equations (18) of a homogeneous chain system and for non-resonance relations

$$\Omega_3 \neq \omega_s = 2\sqrt{\frac{c}{m}}\sin\frac{s\pi}{2(N+1)} \quad \text{where } s = 1,2,3,\ldots,N \tag{24}$$

are in the following form

$$x_k = \sum_{s=1}^{s=N} D_s \cos(\omega_s t + \gamma_s) \sin \frac{ks\pi}{(N+1)} + \frac{\Delta_k(\Omega_3^2)}{\Delta(\Omega_3^2)} \cos(\Omega_3 t + \phi_{03}), \quad k = 1, 2, 3, \ldots, N \tag{25}$$

where D_s and γ_s are integral constants depending on initial conditions.

The previous general solution (25) of the system of ordinary differential equations (18) of a homogeneous chain system and for any resonance relations, when

$$\Omega_{3,res,s} = \omega_s = 2\sqrt{\frac{c}{m}} \sin \frac{s\pi}{2(N+1)} \quad \text{where } s = 1, 2, 3, \ldots, N \tag{26}$$

is not valid for any possible resonance relations.

Consider one of these possible resonance relations between system parameters. Then, take into account general solutions of the system of ordinary differential equations (18) of the homogeneous chain dynamics of the system for one of the possible resonance relations, when

$$\Omega_{3,res,M} = \omega_M = 2\sqrt{\frac{c}{m}} \sin \frac{M\pi}{2(N+1)} \quad \text{where } s = M \tag{27}$$

The general solution for that resonance case is obtained in the following form

$$\begin{aligned} x_k = &\sum_{s=1}^{s=N} \left\langle \frac{1}{|\mathbf{R}|} \sum_{j=1}^{j=N} (-1)^{s+r} |\mathbf{R}_{js}| \left\langle x_{0j} \cos\omega_s t - \frac{\dot{x}_{0j}}{\omega_s} \sin\omega_s t \right\rangle \right\rangle \sin\frac{ks\pi}{(N+1)} \\ &- \frac{1}{2\omega_M^2 \Delta^M(\omega_M^2)} \left[-\sum_{s=1}^{s=N} \left\langle \frac{1}{|\mathbf{R}|} \sum_{j=1}^{j=N} (-1)^{s+r} |\mathbf{R}_{js}| \left\langle \omega_M \frac{d\Delta_j(\Omega_3^2)}{d\Omega_3} \right\rangle_{\Omega_3=\omega_M} \right.\right. \\ &\left.\left. \cdot \left\langle \cos(\phi_{03}) \cos\omega_s t - \frac{\omega_M}{\omega_s} \sin(\phi_{03}) \sin\omega_s t \right\rangle \right\rangle \sin\frac{ks\pi}{(N+1)} \right] \\ &- \frac{1}{2\omega_M^2 \Delta^M(\omega_M^2)} \left[-\sum_{s=1}^{s=N} \left\langle \frac{1}{|\mathbf{R}|} \sum_{j=1}^{j=N} (-1)^{s+r} |\mathbf{R}_{js}| \omega_M \Delta_j(\omega_M^2) \cdot \right.\right. \\ &\left.\left. \cdot \left\langle \cos(\phi_{03}) \cos\omega_s t - \frac{1}{\omega_s} \sin(\phi_{03}) \sin\omega_s t \right\rangle \right\rangle \sin\frac{ks\pi}{(N+1)} \right] \\ &- \frac{1}{2\omega_M^2 \Delta^M(\omega_M^2)} \left[\omega_M \left\langle \frac{d\Delta_k(\Omega_3^2)}{dt} \right\rangle_{\Omega_3=\omega_M} \cos(\omega_M t + \phi_{03}) \right. \\ &\left. - \langle \omega_M t \rangle \Delta_k(\omega_M^2) \sin(\omega_M t + \phi_{03}) \right] \\ &k = 1, 2, 3, \ldots, N \end{aligned} \tag{28}$$

From the obtained expressions (28) for mass particle displacements for where resonance at eigen circular frequency is $\Omega_{3,res,M} = \omega_M$, we can identify a term in the form $\langle \omega_M t \rangle \sin(\omega_M t + \phi_{03})$ in all coordinates displacements of mass particles along a longitudinal direction of chains, which has a tendency to increase linearly with time and for a long time period tends to infinity. This phenomenon is similar to the resonant regimes of the dynamics of a system with one degree of freedom. Then elongation of the mass particle displacements in the resonant regime increases and the system has to quickly change the external excitation circular frequency to return it to a regime of stable multi-frequency oscillations, with bounded elongations.

Theorem 3. In the linear chain system forced dynamics, resonance regimes appear when external excitation circular frequency is equal to one of the eigen circular frequencies. Then the maximum number of resonance regimes is possible and it is equal to the number of d.f. of oscillations of the system. In resonance regimes, elongations of mass particles increase linearly and tend to infinity, but it is not infinite for a short timeframe immediately after the resonant regimes started. Then it is necessary to change the external excitation circular frequency so that it is different from any eigen circular frequencies of free system vibrations. It is recommended to take out the system of the resonant regime as soon as possible. If the system is in a resonant regime for a long period of time, elongations will become too big. Dynamical absorption occurs when the amplitude of a particular solution is equal to zero:

$$C_{(3)k}(\Omega_3^2) = \frac{\Delta_k(\Omega_3^2)}{\Delta(\Omega_3^2)} = 0, \text{ for } k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \tag{29}$$

A paradox that for some values of external excitation frequency of applied force to the certain mass particle, the mass particle is in a forced state of dynamical absorption, seems to be very interesting. Hence, this mass particle does not have a forced component with external excitation frequency and vibrates only with eigen circular frequencies excited by initial perturbation of the equilibrium state. If there is no initial perturbation of the equilibrium state, the third mass particle rests, as a paradox, continuously when it is under the action of this external one frequency excitation.

As homogeneous chains are used as filters, it is useful to consider kinematical excitation at the end of mechanical or electrical chain systems. Using the results from Hedrih (Stevanović) (2006b), kinematical (rheonomic) excitation may be defined by displacement of the left end of a spring in the form: $x_0 = A_0 \cos \Omega t$. Then this task is defined as: $\frac{m}{c}\ddot{x}_k =$

$-(x_k - x_{k-1}) + (x_{k+1} - x_k)$, $k = 1, 2, \ldots, N$ with boundary conditions: $x_0 = A_0 \cos \Omega t$ and $x_{N+1} = 0$. All mass particles will oscillate in a forced regime by the same external kinematical excitation circular frequency Ω, but with different amplitude (Rašković, 1952, 1974; Hedrih (Stevanović), 1991): $x_k = A_k \cos \Omega t$. Suppose that amplitude A_k is taken in the form $A_k = D \sin(k\varphi + \alpha)$. From boundary conditions it is easy to obtain the following relations:

$$A_0 = D \sin \alpha, \quad \alpha = s\pi - (N+1)\varphi \quad \text{and}$$
$$\frac{A_k}{A_0} = \frac{(-1)^{s+1} D \sin(N+1-k)\varphi}{(-1)^{s+1} D \sin(N+1)\varphi} = \frac{\sin(N+1-k)\varphi}{\sin(N+1)\varphi} = \eta_{dk}$$
$$\text{for } k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \tag{30}$$

η_{dk} is the dynamic factor for each mass particle of the homogeneous chain displacements (Rašković, 1974). Where chains are excited kinematically by $x_0 = A_0 \cos \Omega t$, expressions for mass particle displacements are

$$x_k = A_k \cos \Omega t = A_0 \frac{\sin(N+1-k)\varphi}{\sin(N+1)\varphi} \cos \Omega t \tag{31}$$
$$\text{for } k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$$

4. Analysis of amplitude frequency graphs for homogeneous chain dynamics excited by one frequency external excitation applied to athird mass particle

Numerical analysis of kinetic frequency parameters of the homogeneous chain dynamics of a system with 11 d.f. for free and forced vibration analytically described in previous sections was performed by two sets of numerical data of system parameters. The first numerical data are for the system at macro level, each mass particle has the same mass, $m_k = m = 1[\text{kg}]$, and all springs have the same coefficient of rigidity $c_k = c = 1\,[\text{N/m}] = 1\,[\text{kg/sec}^2]$.

The second numerical data are for the system at nano level, the mass particles are with the same masses $m_k = m = 137,842 \cdot 10^{-12}[\text{kg}]$, and all springs have the same coefficient of rigidity $c_k = c = 246,75\,[\text{N/m}]$.

In Figure 2, two characteristic frequency curves with 11 roots – a homogeneous chain system eigen characteristic number (square values of eigen circular frequencies) for free vibrations are presented. In Figure 2(a) a characteristic frequency curve with 11 roots for a mechanical chain system with 11 mass particles at nano level is presented. These roots are x1 = 0,0675; x2 = 0,27; x3 = 0,585; x4 = 1,01; x5 = 1,48; x6 = 2,005; x7 = 2,515 x8 = 3,0; x9 = 3,41; x10 = 3,735; x11 = 3,93.

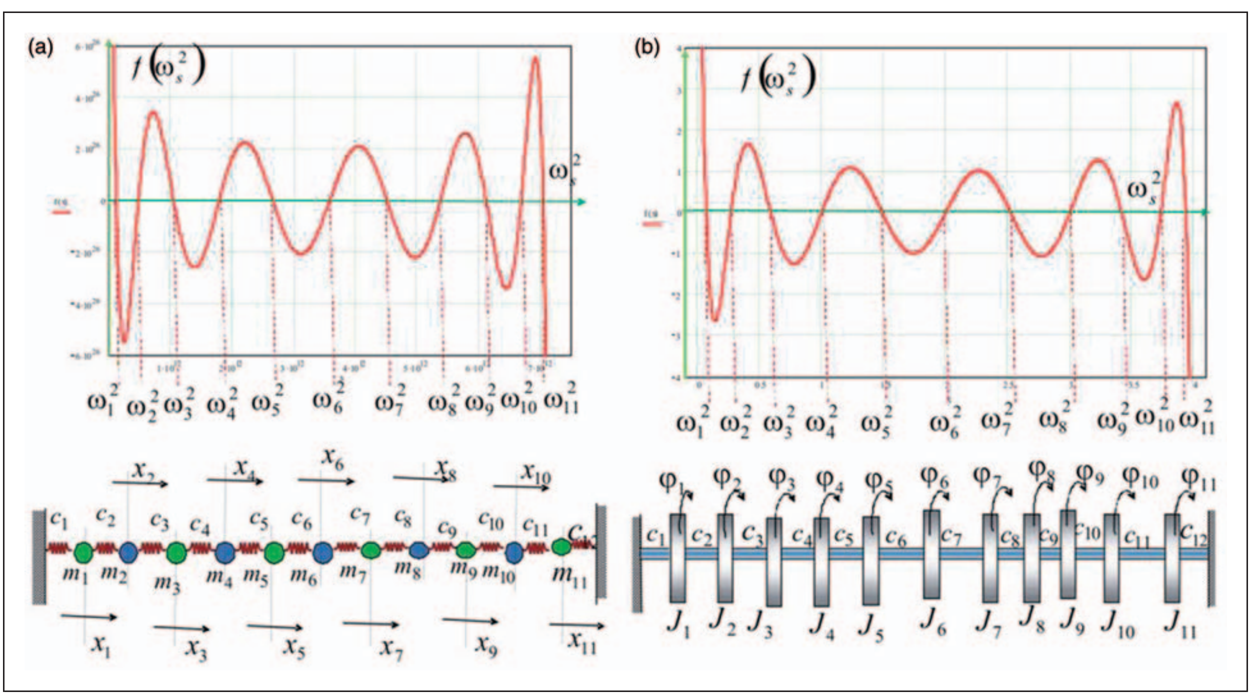

Figure 2. Characteristic frequency curves with 11 roots – homogeneous chain system eigen characteristic numbers (square values of eigen circular frequencies) for free vibrations. (a) for mechanical chain system with 11 mass particles; (b) for mechanical chain torsion system with 11 disks on the shaft

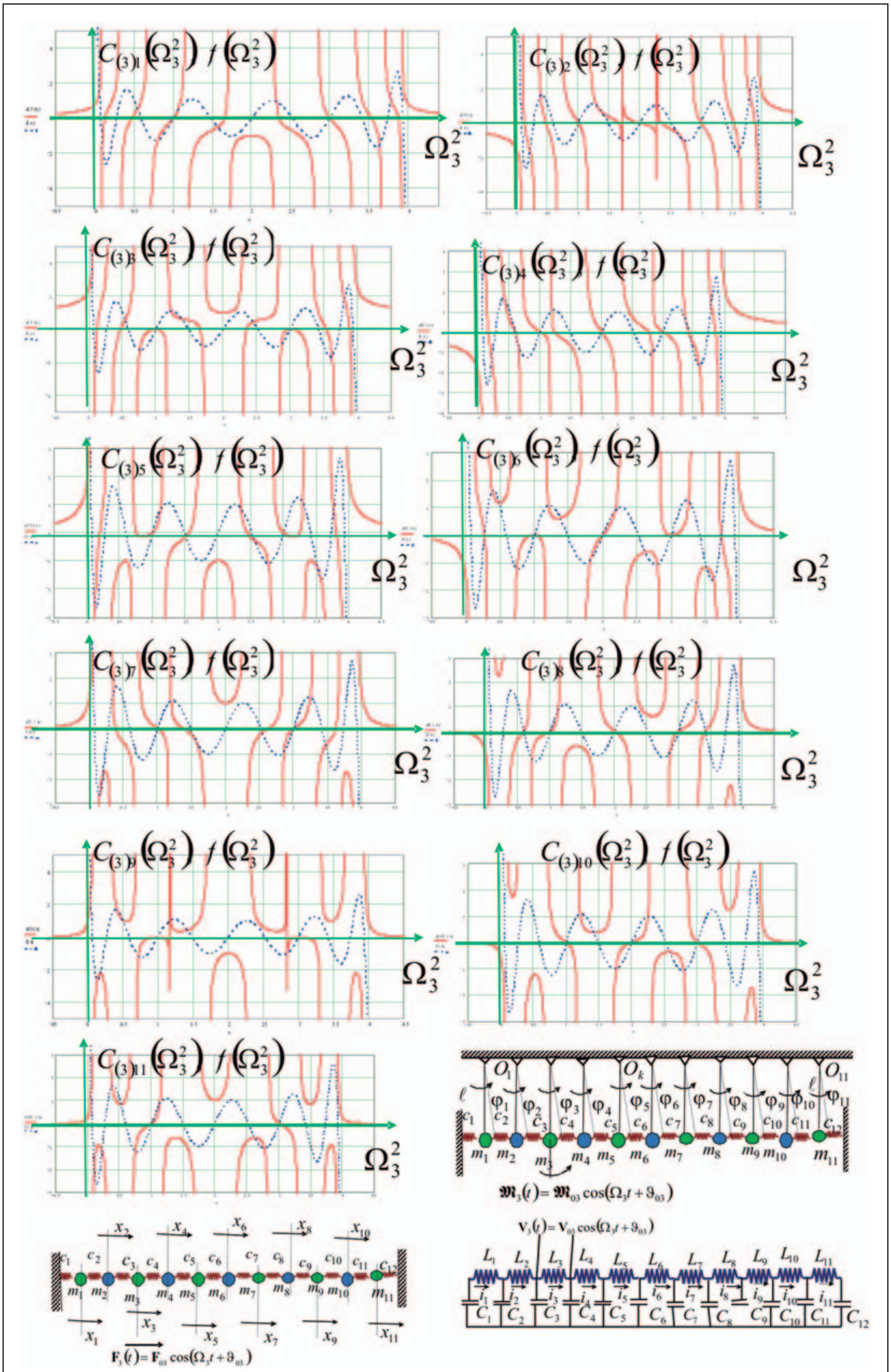

Figure 3. Eleven amplitude-frequency resonant curves of a homogenous chain system at nano level with 11 degrees of freedom for 11 generalized coordinate forced vibrations under the action of one frequency: external excitation applied to the third mass particle in chain or external electric voltage applied to the third electric sub-circuit to the coil inductor and capacitor or external mechanical couple applied to the third pendulum in the chain's multi-pendulum system

In Figure 2(b) a characteristic frequency curve with 11 roots for a mechanical chain torsion system with 11 disks on the shaft is presented by using an analogy between systems (presented in Figure 1(a) and (b)). Numerical calculation was done by data for a chain system with mass particles on the macro level and by analogy mass particles with the same masses $m_k = m = 1[\text{kg}]$, and all springs have the same coefficient of rigidity $c_k = c = 1\,[\text{N/m}] = 1\left[\text{kg/sec}^2\right]$; by using mathematical analogy and phenomenological mapping between parameters and coordinates of the two chain systems and corresponding analogous axial mass inertia moments of 11 disks $J_k = J = 1[\text{kgm}^2]$ and coefficients of torsion rigidities of the parts' shaft between disks $c_k = c = 1\,[\text{N/m}] = 1\left[\text{kg/sec}^2\right]$, Figure 2(b) was produced. Eigen characteristic numbers as well as eigen circular frequencies for a mechanical chain torsion system with 11 disks on the shaft have the same corresponding characteristic numbers as a mechanical chain system with 11 mass particles.

In Figure 3, 11 amplitude-frequency resonant curves of a homogenous chain system at nano level with 11 d.f. for 11 generalized coordinate forced vibrations of mass particles displacements and perturbation positions, in relation to equilibrium, under the action of one frequency external excitation applied to the third mass particle in the chain are presented in the function of external excitation square frequency.

Using ideas and theory of mathematical phenomenology and mathematical analogy, the same 11 amplitudes – frequency resonant curves, that are presented in Figure 3, can also, be used for systems with eleven degree of freedom presented in Figure 1c (chain multi pendulum system) and Figure 1d (chain electrical circuit system) under corresponding analogous kinetic parameters of systems and analogous parameters of external excitation loads, as well as corresponding analogous boundary and initial conditions.

From the 11 amplitude-frequency curves presented in Figure 3 it can be seen that in a chain system with 11 d.f., for forced dynamics excited by one frequency external excitation applied to the third mass particle, the appearance of 11 resonant regimes at values of eigen circular frequencies is possible. In each of the 11 amplitude-frequency curves at some values of eigen circular frequencies the curves possess vertical asymptotes. When amplitude-frequency curves take zero values, then the amplitude of particular solutions are equal to zero, and the corresponding circular frequencies of the external excitation correspond to the dynamical absorption regime, and the corresponding mass particle is at forced peace or vibrates only with eigen circular frequencies, depending on initial conditions, but there are no vibrations with the external one frequency excitation circular frequency, as a paradox, when external excitation is applied to this mass particle.

From the 11 amplitude-frequency curves presented in Figure 3 it can be seen that dynamical absorption appears in the forced vibrations of:

- the first mass particle at eight circular frequencies of external one frequency excitation;
- the second mass particle at nine circular frequencies of external one frequency excitation;
- the third mass particle at six circular frequencies of external one frequency excitation;
- the fourth mass particle at nine circular frequencies of external one frequency excitation;
- the fifth mass particle at four circular frequencies of external one frequency excitation;
- the sixth mass particle at three circular frequencies of external one frequency excitation;
- the seventh mass particle at six circular frequencies of external one frequency excitation;
- the eighth mass particle at five circular frequencies of external one frequency excitation;
- the ninth mass particle at zero circular frequencies of external one frequency excitation;
- the tenth mass particle at three circular frequencies of external one frequency excitation;
- the eleventh mass particle at two circular frequencies of external one frequency excitation.

The appearance of one or more regimes of dynamic absorptions at the resonant frequencies is possible, so it is conceivable that some mass particles in the chain are in the regime of resonant oscillations, while other mass particles are in the regime of dynamic absorption. This is evident for example in Figure 3 where the amplitude-frequency curves of each mass particle in the chain are presented, while the amplitude-frequency curve possesses roots at the same point as the characteristic frequency graph.

5. Non-homogeneous model of chain system forced vibrations

Let us consider a non-homogeneous model of chain system kinetic parameters. Corresponding matrix $\mathbf{A}$ of inertia elements, for a system with 11 d.f., is a diagonal quadratic of 11th order and generalized external excitation forces $\{Q_3\}$ are in the following forms:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} m_1 & \dots & m_{11} \end{pmatrix} I, \\ (Q_3) &= \begin{pmatrix} 0 & 0 & F_{03} & 0 & \dots & 0 \end{pmatrix} \cos \Omega_3 t \end{aligned} \tag{32}$$

Where I is the unit diagonal matrix of 11th order.

Without losing generality, we have adopted the single-frequency external excitation attached to the third mass particle in the chain.

Matrix $\mathbf{C}$ of the rigidity coefficients of the spring elements for a chain system with 11 d.f. is a tridiagonal matrix of 11th order with bandwidth three in the following form

$$C=\begin{pmatrix} c_1+c_2 & -c_2 & & & & & & & & & \\ -c_2 & (c_2+c_3) & -c_3 & & & & & & & & \\ & -c_3 & (c_3+c_4) & -c_4 & & & & & & & \\ & & -c_4 & (c_4+c_5) & -c_5 & & & & & & \\ & & & -c_5 & (c_5+c_6) & -c_6 & & & & & \\ & & & & -c_6 & (c_6+c_7) & -c_7 & & & & \\ & & & & & -c_7 & (c_7+c_8) & -c_8 & & & \\ & & & & & & -c_8 & (c_8+c_9) & -c_9 & & \\ & & & & & & & -c_9 & (c_9+c_{10}) & -c_{10} & \\ & & & & & & & & -c_{10} & c_{10}+c_{11} & -c_{11} \\ & & & & & & & & & -c_{11} & c_{11}+c_{12} \end{pmatrix} \tag{33}$$

The frequency equation expressed by previous matrix $\mathbf{A}$ and $\mathbf{C}$ is in the form of the following determinant

$$f(\omega^2)=\left|\mathbf{C}-\omega^2\mathbf{A}\right|=0 \tag{34}$$

or in developed form

$$f(\omega^2)=\begin{vmatrix} (c_1+c_2)-\omega^2 m_1 & -c_2 & & & & & & & & & \\ -c_2 & (c_2+c_3)-\omega^2 m_2 & -c_3 & & & & & & & & \\ & -c_3 & (c_3+c_4)-\omega^2 m_3 & -c_4 & & & & & & & \\ & & -c_4 & (c_4+c_5)-\omega^2 m_4 & -c_5 & & & & & & \\ & & & -c_5 & (c_5+c_6)-\omega^2 m_5 & -c_6 & & & & & \\ & & & & -c_6 & (c_6+c_7)-\omega^2 m_6 & -c_7 & & & & \\ & & & & & -c_7 & (c_7+c_8)-\omega^2 m_7 & -c_8 & & & \\ & & & & & & -c_8 & (c_8+c_9)-\omega^2 m_8 & -c_9 & & \\ & & & & & & & -c_9 & (c_9+c_{10})-\omega^2 m_9 & -c_{10} & \\ & & & & & & & & -c_{10} & (c_{10}+c_{11})-\omega^2 m_{10} & -c_{11} \\ & & & & & & & & & -c_{11} & (c_{11}+c_{12})-\omega^2 m_{11} \end{vmatrix}_{=0} \tag{35}$$

The roots of the previous frequency equation – the characteristic equation – are the eigen characteristic numbers of the square of the eigen circular frequencies. The system of ordinary differential equations of mass particle forced dynamics is in the following matrix form

$$\mathbf{A}\{\ddot{x}\}+\mathbf{C}\{x\}=\left\{Q\right\}=\{F_0\}\cos\Omega t \tag{36}$$

or in the form with proposed particular solutions

$$\begin{aligned}&\left(\ddot{x}_1 \quad \ddot{x}_2 \quad \ddot{x}_3 \quad \ddot{x}_4 \quad \ldots \quad \ddot{x}_{11}\right)\mathbf{A}'\\ &+\left(x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_{11}\right)\mathbf{C}'\\ &=\left(0 \quad 0 \quad F_{03} \quad 0 \quad \ldots \quad 0\right)\cos\Omega_3 t\end{aligned}$$

Where

$$\begin{aligned}&\left(x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_{11}\right)\\ &=\left(C_1 \quad C_2 \quad C_3 \quad C_4 \quad \ldots \quad C_{11}\right)\cos\Omega_3 t\end{aligned}$$

and it follows that

$$\begin{pmatrix} C_1 & C_2 & C_3 & C_4 & \dots & C_{11} \end{pmatrix}\left(\mathbf{C}' - \Omega_3^2\mathbf{A}'\right) = \begin{pmatrix} 0 & 0 & F_{03} & 0 & \dots & 0 \end{pmatrix} \tag{37}$$

The determinant of the previous system is in the form

$$\Delta\left(\Omega_3^2\right) = \left|\mathbf{C} - \Omega_3^2\mathbf{A}\right| \neq 0 \tag{38}$$

and must be different from zero. Then it must be $\Omega_3^2 \neq \omega_s^2$, $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

If the determinant of the previous system is equal to zero: $f\left(\Omega_3^2\right) = \Delta\left(\Omega_3^2\right) = \left|\mathbf{C} - \Omega_3^2\mathbf{A}\right| = 0$ for $\Omega_3^2 = \omega_s^2$, $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, then the external excitation circular frequency is the resonant frequency, as we show for homogeneous chain forced dynamics. Now it must be seen if this holds for a non-homogeneous system.

The determinant of the chain system with 11 d.f. is in the following developed form

Amplitude $C_{(3)1}\left(\Omega_3^2\right)$ of particular solution for the first mass particle in the chain forced vibration displacement under the external one frequency excitation applied to the third mass particle in chain, is in the form

$$C_{(3)1}\left(\Omega_3^2\right) = \frac{\Delta_1\left(\Omega_3^2\right)}{\Delta\left(\Omega_3^2\right)}, \quad \text{for } \Omega_3^2 \neq \omega_s^2, \quad s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \tag{41}$$

The particular solution for the first mass particle in chains, when one frequency external excitation is applied to the third mass particle in the chain, is in the form:

$$x_1\left(t, \Omega_3^2\right) = C_{(3)1}\left(\Omega_3^2\right)\cos\Omega_3 t, \quad \text{for } \Omega_3^2 \neq \omega_s^2, \quad s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \tag{42}$$

$$f(\Omega_3^2) = \Delta(\Omega_3^2) = \begin{vmatrix} (c_1+c_2)-\Omega_3^2 m_1 & -c_2 & & & & & & & & & \\ -c_2 & (c_2+c_3)-\Omega_3^2 m_2 & -c_3 & & & & & & & & \\ & -c_3 & (c_3+c_4)-\Omega_3^2 m_3 & -c_4 & & & & & & & \\ & & -c_4 & (c_4+c_5)-\Omega_3^2 m_4 & -c_5 & & & & & & \\ & & & -c_5 & (c_5+c_6)-\Omega_3^2 m_5 & -c_6 & & & & & \\ & & & & -c_6 & (c_6+c_7)-\Omega_3^2 m_6 & -c_7 & & & & \\ & & & & & -c_7 & (c_7+c_8)-\Omega_3^2 m_7 & -c_8 & & & \\ & & & & & & -c_8 & (c_8+c_9)-\Omega_3^2 m_8 & -c_9 & & \\ & & & & & & & -c_9 3 & (c_9+c_{10})-\Omega_3^2 m_9 & -c_{10} & \\ & & & & & & & & -c_{10} & (c_{10}+c_{11})-\Omega_3^2 m_{10} & -c_{11} \\ & & & & & & & & & -c_{11} & (c_{11}+c_{12})-\Omega_3^2 m_{11} \end{vmatrix} \tag{39}$$

and must to be different from zero, for $\Omega_3^2 \neq \omega_s^2$, $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

To obtain amplitudes $C_{(3)k}\left(\Omega_3^2\right)$ of particular solutions, it is necessary to find 11 sub-determinants $\Delta_k\left(\Omega_3^2\right)$. Corresponding sub-determinants $\Delta_k\left(\Omega_3^2\right)$, $k = 1, 2, 3, 4, \dots, 11$: it is possible to obtain by substituting into the determinate of system $\Delta\left(\Omega_3^2\right)$ expressed in (39) the corresponding k-th column by column $\{\delta_{3k}F_{03}\}$, containing non-zero element F_{03}. Where F_{03} is the amplitude of external excitation force applied to the third mass particle of the chain system. For the first sub-determinant $\Delta_1\left(\Omega_3^2\right)$, the following can be written

This amplitude (41) of the particular solution is the function of the external excitation circular frequency force. When, the amplitude-frequency graph, as a function of external excitation circular frequency, has the value of one of the eigen circular frequency values, then the graph has asymptotes and branches that tend to infinity. When external excitation frequency takes the values of eigen circular frequency of free oscillations, amplitudes of particular solutions for forced vibrations asymptotically tend to infinity and the vibration state corresponds to a resonant state in which elongations

$$\Delta_1(\Omega_3^2) = F_{03} \begin{vmatrix} -c_2 & & & & & & & & & \\ (c_2+c_3)-\Omega_3^2 m_2 & -c_3 & & & & & & & & \\ & -c_4 & (c_4+c_5)-\Omega_3^2 m_4 & -c_5 & & & & & & \\ & & -c_5 & (c_5+c_6)-\Omega_3^2 m_5 & -c_6 & & & & & \\ & & & -c_6 & (c_6+c_7)-\Omega_3^2 m_6 & -c_7 & & & & \\ & & & & -c_7 & (c_7+c_8)-\Omega_3^2 m_7 & -c_8 & & & \\ & & & & & -c_8 & (c_8+c_9)-\Omega_3^2 m_8 & -c_9 & & \\ & & & & & & -c_9 & (c_9+c_{10})-\Omega_3^2 m_9 & -c_{10} & \\ & & & & & & & -c_{10} & c_{10}+c_{11}-\Omega_3^2 m_{10} & -c_{11} \\ & & & & & & & & -c_{11} & c_{11}+c_{12}-\Omega_3^2 m_{11} \end{vmatrix} \tag{40}$$

increase and tend to infinity with increasing time duration of this resonant state.

Amplitude $C_{(3)k}\left(\Omega_3^2\right)$ of the particular solution for k-th mass particle in the chain forced vibration displacement under the external one frequency excitation applied to the third mass particle in chain, is in the form

$$C_{(3)k}\left(\Omega_3^2\right)=\frac{\Delta_{(3)k}\left(\Omega_3^2\right)}{\Delta\left(\Omega_3^2\right)},\quad \text{for } \Omega_3^2\neq\omega_s^2, \\ s=1,2,3,4,5,6,7,8,9,10,11 \tag{43}$$

The particular solution for k-th mass particle in chains, when one frequency external excitation is applied to the third mass particle in the chain, is in the form

$$x_k\left(t,\Omega_3^2\right)=C_{(3)k}\left(\Omega_3^2\right)\cos\Omega_3 t \tag{44}$$

Corresponding determinants for obtaining amplitudes of the particular solution for the third mass particle and the ninth mass particle in the chain are in the following form

The general solution of the system of ordinary differential equations (37) describing forced vibrations of a non-homogeneous chain under the external one frequency excitation applied to the third mass particle in the chain is in the form

$$x_k\left(t,\Omega_3^2\right)=\sum_{s=1}^{s=N}K_{Nk}^{(s)}\left(\omega_s^2\right)\langle A_s\cos\omega_s t+B_s\sin\omega_s t\rangle \\ +\frac{\Delta_k\left(\Omega_3^2\right)}{\left(\omega_M^2-\Omega_3^2\right)\Delta^{(N-1)}\left(\Omega_3^2\right)}\cos\Omega_3 t \\ \text{for } k=1,2,3,4,5,6,7,8,9,10,11 \text{ and} \\ \text{for } \Omega_3^2\neq\omega_s^2,\ s=1,2,3,4,5,6,7,8,9,10,11. \tag{47}$$

in which $K_{Nk}^{(s)}\left(\omega_s^2\right)$, $s=1,2,3,4,5,6,7,8,9,10,11$ are cofactors of frequency determinant (32), or elements of modal chain matrix

$$\mathbf{R}=\left(K_{Nk}^{(s)}\right)_{\rightarrow\ s=1,2,3,\ldots,N}^{\downarrow\quad k=1,2,3,\ldots,N}$$

$$\Delta_3\left(\Omega_3^2\right)=F_{03}\begin{vmatrix}(c_1+c_2)-\Omega_3^2m_1 & -c_2 & & & & & & & & \\ -c_2 & (c_2+c_3)-\Omega_3^2m_2 & & & & & & & & \\ & & (c_4+c_5)-\Omega_3^2m_4 & -c_5 & & & & & & \\ & & -c_5 & (c_5+c_6)-\Omega_3^2m_5 & -c_6 & & & & & \\ & & & -c_6 & (c_6+c_7)-\Omega_3^2m_6 & -c_7 & & & & \\ & & & & -c_7 & (c_7+c_8)-\Omega_3^2m_7 & -c_8 & & & \\ & & & & & -c_8 & (c_8+c_9)-\Omega_3^2m_8 & -c_9 & & \\ & & & & & & -c_9 & (c_9+c_{10})-\Omega_3^2m_9 & -c_{10} & \\ & & & & & & & -c_{10} & (c_{10}+c_{11})-\Omega_3^2m_{10} & -c_{11} \\ & & & & & & & & -c_{11} & c_{11}+c_{12}-\Omega_3^2m_{11}\end{vmatrix} \tag{45}$$

$$\Delta_9\left(\Omega_3^2\right)=F_{03}\begin{vmatrix}(c_1+c_2)-\Omega_3^2m_1 & -c_2 & & & & & & & & & \\ -c_2 & (c_2+c_3)-\Omega_3^2m_2 & -c_3 & & & & & & & & \\ & & -c_4 & (c_4+c_5)-\Omega_3^2m_4 & -c_5 & & & & & & \\ & & & -c_5 & (c_5+c_6)-\Omega_3^2m_5 & -c_6 & & & & & \\ & & & & -c_6 & (c_6+c_7)--\Omega_3^2m_6 & -c_7 & & & & \\ & & & & & -c_7 & (c_7+c_8)-\Omega_3^2m_7 & -c_8 & & & \\ & & & & & & -c_8 & (c_8+c_9)-\Omega_3^2m_8 & & & \\ & & & & & & & -c_9 & & -c_{10} & \\ & & & & & & & & & c_{10}+c_{11}-\Omega_3^2m_{10} & -c_{11} \\ & & & & & & & & & -c_{11} & c_{11}+c_{12}-\Omega_3^2m_{11}\end{vmatrix} \tag{46}$$

with a column containing eigen vectors, $\Delta(\Omega_3^2) = (\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)$ and A_s and B_s, $s = 1,2,3,4,5,6,7,8,9,10,11$ are integral constants defined by the initial conditions. By using the initial conditions at initial moment $t = 0$, mass particles have perturbation in relation to equilibrium positions $x_k(0) = x_{0k}$ and initial velocities: $\dot{x}_k(0) = \dot{x}_{0k}$. Based on the accepted initial conditions it is correct for integral constants A_s and B_s, $s = 1,2,3,4,5,6,7,8,9,10,11$ to write the following expressions

$$A_s = \frac{1}{|\mathbf{R}|}\sum_{j=1}^{j=N}(-1)^{j+s}|\mathbf{R}_{js}|\left\langle x_{0j} - \frac{\Delta_j(\Omega_3^2)}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\right\rangle$$
$$B_s = -\frac{1}{|\mathbf{R}|}\sum_{j=1}^{j=N}(-1)^{j+s}|\mathbf{R}_{js}|\left\langle \frac{\dot{x}_{0j}}{\omega_s} + \frac{\Omega_3}{\omega_s}\frac{\Delta_j(\Omega_3^2)}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\right\rangle$$
$$s = 1,2,3,4,5,6,7,8,9,10,11 \tag{48}$$

where $|\mathbf{R}_{js}|$ are corresponding co-factors of modal chain matrix

$$\mathbf{R} = \left(K_{Nk}^{(s)}\right)^{\downarrow\ k=1,2,3,\ldots,N}_{\rightarrow\ s=1,2,3,\ldots,N}.$$

The particular solution of the system of ordinary differential equations (37) for forced vibrations and known initial conditions, describing chain mass particle forced non-resonant vibration displacements, is in the following form

$$\begin{aligned} x_k(t,\Omega_3^2) &= \frac{1}{|\mathbf{R}|}\sum_{s=1}^{s=N}\left[K_{Nk}^{(s)}(\omega_s^2)\sum_{j=1}^{j=N}(-1)^{j+s}|\mathbf{R}_{js}|\right.\\ &\quad\left.\cdot\left\langle x_{0j}\cos\omega_s t + \frac{\dot{x}_{0j}}{\omega_s}\sin\omega_s t\right\rangle\right]\\ &\quad - \frac{1}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\frac{1}{|\mathbf{R}|}\\ &\quad\times\sum_{s=1}^{s=N}\left[K_{Nk}^{(s)}(\omega_s^2)\sum_{j=1}^{j=N}(-1)^{j+s}\right.\\ &\quad\left.\cdot|\mathbf{R}_{js}|\Delta_j(\Omega_3^2)\left\langle\cos\omega_s t + \frac{\Omega_3}{\omega_s}\sin\omega_s t\right\rangle\right]\\ &\quad + \frac{\Delta_k(\Omega_3^2)}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\cos\Omega_3 t\\ &\quad \text{for } k = 1,2,3,4,5,6,7,8,9,10,11 \text{ and}\\ &\quad \text{for } \Omega_3^2 \neq \omega_s^2,\ s = 1,2,3,4,5,6,7,8,9,10,11. \end{aligned} \tag{49}$$

The first series of terms in the previous expression (49) is the particular solution for free vibrations and known initial conditions in the following form

$$\begin{aligned} x_k(t,\Omega_3^2) &= \frac{1}{|\mathbf{R}|}\sum_{s=1}^{s=N}\left[K_{Nk}^{(s)}(\omega_s^2)\sum_{j=1}^{j=N}(-1)^{j+s}|\mathbf{R}_{js}|\right.\\ &\quad\left.\cdot\left\langle x_{0j}\cos\omega_s t + \frac{\dot{x}_{0j}}{\omega_s}\sin\omega_s t\right\rangle\right]\\ &\quad \text{for } k = 1,2,3,4,5,6,7,8,9,10,11 \end{aligned} \tag{50}$$

and contain only a set of eigen circular frequencies of free chain vibrations, ω_s, $s = 1,2,3,4,5,6,7,8,9,10,11$. Free vibrations of each mass particle in a chain with 11 d.f. are in the multi (11) frequency vibration regime.

The particular solution (49) for mass particle forced vibrations and known initial conditions contains, in the first part, terms from expression (50) for free vibrations and with 11 eigen circular frequencies, and in the second part, terms, now denoted by $x_{k(3,4)}(t,\Omega_3^2)$, depending on the set of eigen circular frequencies of free vibrations and on external excitation circular frequency $\Omega_3 \neq \omega_s$.

When external excitation circular frequency Ω_3 is equal to one of the eigen circular frequencies of free vibrations $\Omega_3 = \omega_M$, from the set ω_s, $\Omega_3^2 = \omega_M^2 = \omega_s^2$, $M = s = 1,2,3,4,5,6,7,8,9,10,11$ terms from solution (49) denoted by $x_{k(3,4)}(t,\Omega_3^2)$ take undefined value $[x_{k(3,4)}(t,\Omega_3^2)]_{\Omega_3=\omega_M} = \frac{0}{0}$ and it is necessary to apply L'Hôpital's rule to find the limit of this expression value $[x_{k(3,4)}(t,\Omega_3^2)]_{\Omega_3=\omega_M} = \lim_{\Omega_3\to\omega_M} x_{k(3,4)}(t,\Omega_3^2)$ (see Hedrih (Stevanović), 2006b):

$$\begin{aligned} &x_{k(3,4)}(t,\Omega_3^2)\\ &= \lim_{\Omega_3\to\omega_M}\left\{-\frac{1}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\frac{1}{|\mathbf{R}|}\sum_{s=1}^{s=N}\left[K_{Nk}^{(s)}(\omega_s^2)\right.\right.\\ &\quad\left.\cdot\left\langle\sum_{j=1}^{j=N}(-1)^{j+s}|\mathbf{R}_{js}|\Delta_j(\Omega_3^2)\left\langle\cos\omega_s t + \frac{\Omega_3}{\omega_s}\sin\omega_s t\right\rangle\right\rangle\right]\\ &\quad\left. + \frac{\Delta_k(\Omega_3^2)}{(\omega_M^2 - \Omega_3^2)\Delta^{(N-1)}(\Omega_3^2)}\cos\Omega_3 t\right\}\\ &\text{for } k = 1,2,3,4,5,6,7,8,9,10,11 \text{ and}\\ &\text{for } \Omega_3^2 = \omega_s^2,\ s = 1,2,3,4,5,6,7,8,9,10,11 \end{aligned} \tag{51}$$

After obtaining the limit of this expression value $[x_{k(3,4)}(t,\Omega_3^2)]_{\Omega_3=\omega_M} = \lim_{\Omega_3\to\omega_M} x_{k(3,4)}(t,\Omega_3^2)$ for a resonant case $\Omega_3 = \omega_M$ of 11 possible cases $\Omega_3^2 = \omega_M^2 = \omega_s^2$, $M = s = 1,2,3,4,5,6,7,8,9,10,11$, the particular solution of the system of ordinary differential equations

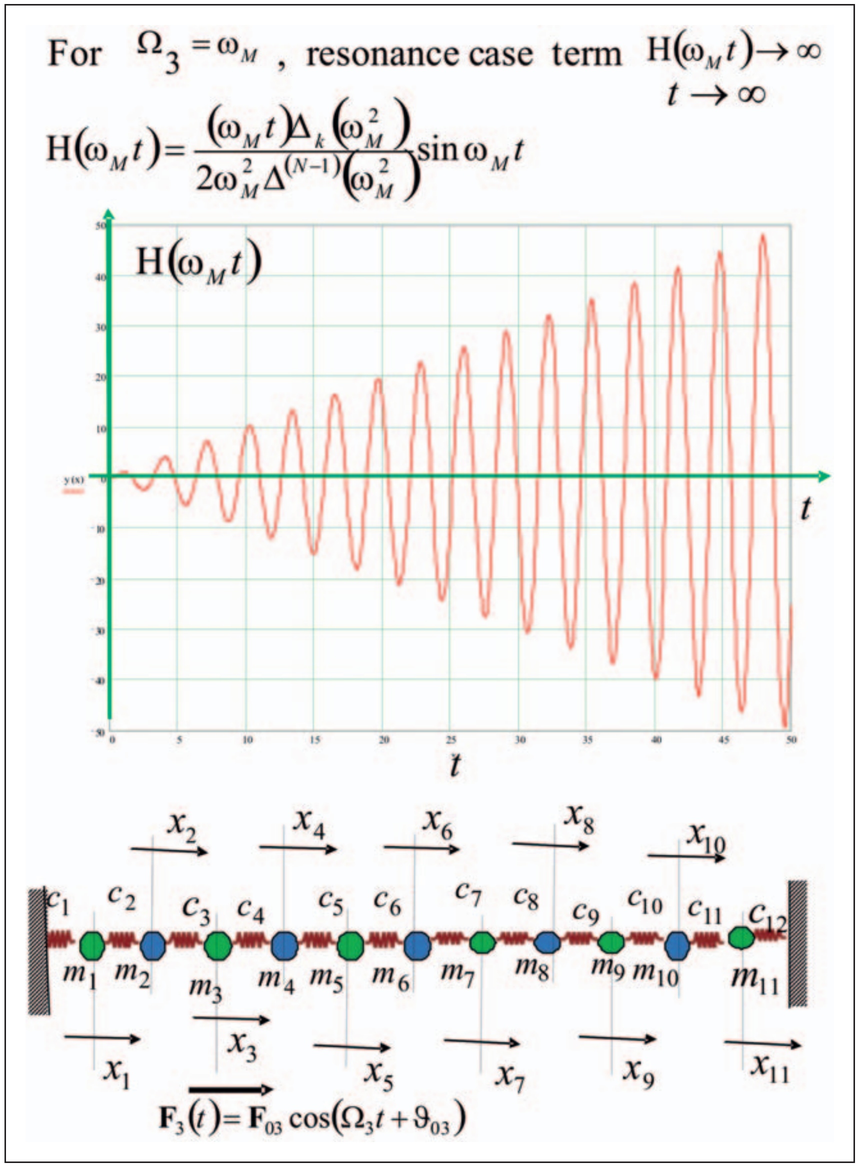

Figure 4. Time history graph of the resonance summand H($\omega_M t$) in expressions of angular coordinate of disks in chain torsion system (or of mass particle displacement in chain system) linear increasing with time

(37) for forced vibrations and known initial conditions, describing chain mass particle forced resonant vibration displacements, is in the following form:

$$
\begin{aligned}
x_{k(3,4)}\left(t,\Omega_3^2\right) = &-\frac{1}{-2\omega_M \Delta^{(N-1)}\left(\omega_M^2\right)} \\
&\cdot\left\{\frac{1}{|\mathbf{R}|}\sum_{s=1}^{s=N} K_{Nk}^{(s)}\left(\omega_s^2\right)\sum_{j=1}^{j=N}(-1)^{j+s}\left|\mathbf{R}_{js}\right|\frac{d\Delta_j\left(\Omega_3^2\right)}{d\Omega_3}\bigg|_{\Omega_3=\omega_M}\right. \\
&\cdot\left\langle\cos\omega_s t+\frac{\omega_M}{\omega_s}\sin\omega_s t\right\rangle+\frac{1}{|\mathbf{R}|}\sum_{s=1}^{s=N}K_{Nk}^{(s)}\left(\omega_s^2\right) \\
&\times\sum_{j=1}^{j=N}(-1)^{j+s}\left|\mathbf{R}_{js}\right|\Delta_j\left(\omega_M^2\right)\left\langle\frac{1}{\omega_s}\sin\omega_s t\right\rangle \\
&\left.+\frac{d\Delta_k\left(\Omega_3^2\right)}{d\Omega_3}\bigg|_{\Omega_3=\omega_M}\cos\omega_M t - t\Delta_k\left(\omega_M^2\right)\sin\omega_M t\right\}
\end{aligned}
$$

for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ and

for $\Omega_3^2 = \omega_s^2,\ s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ (52)

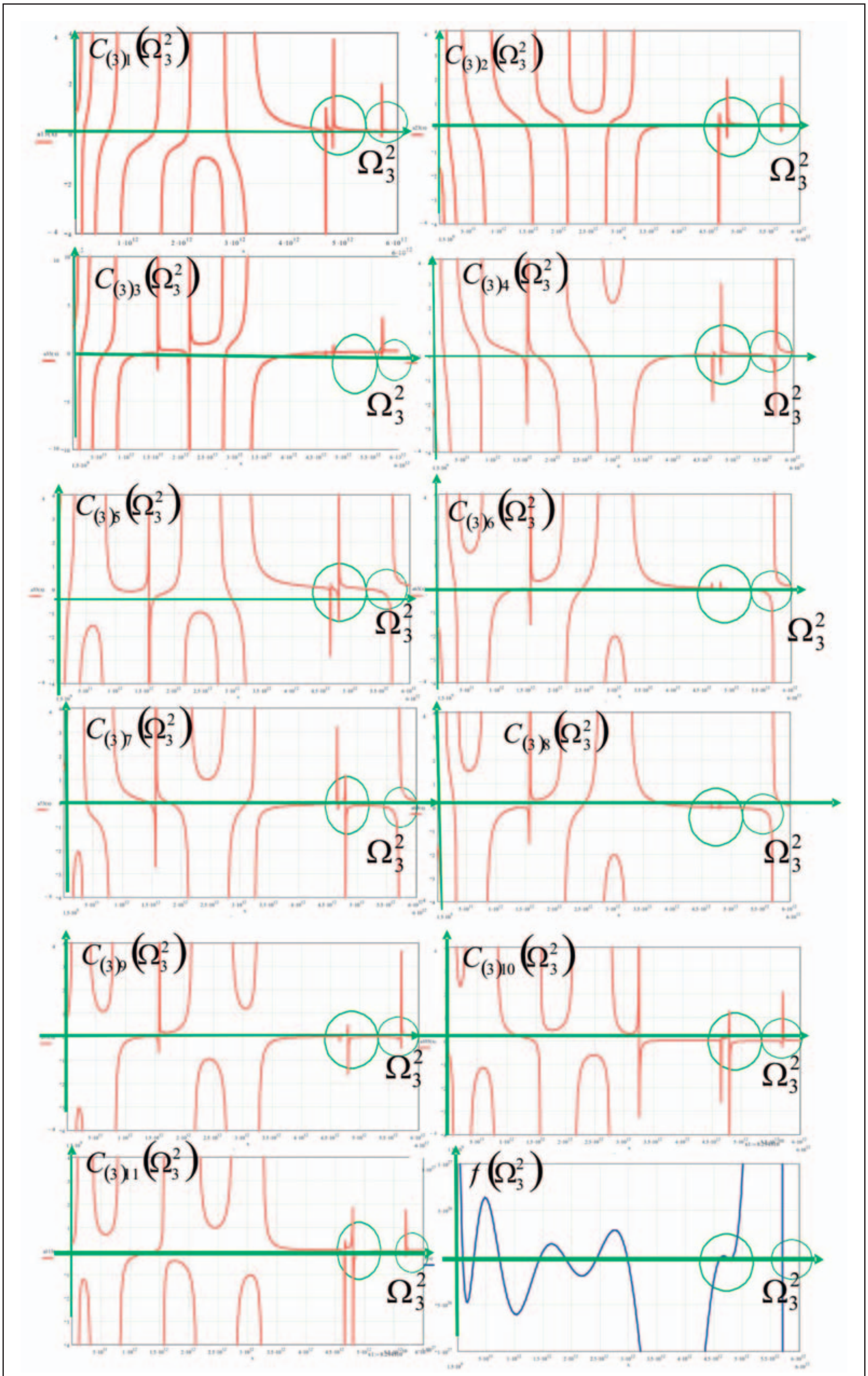

Figure 5. Eleven amplitude-frequency resonant curves of a non-homogeneous chain system for data on the nano level (example of molecules of a mouse *zona pellucida*) with 11 degrees of freedom for forced vibrations under the action of one frequency: external excitation applied to the third mass particle in chain ors electric voltage applied to the third electric sub-circuit to the coil inductance and capacitor or mechanical couple applied to the third pendulum in a multi-pendulum chain system. Circles in figures denote frequency intervals with close asymptotes on resonant frequencies

The previous expression (52) for mass particle displacements in a chain (or in mathematical analogy an angular coordinate of disk rotation about a shaft axis in mechanical torsion chain, Figure 1(c)) in a resonant case when $\Omega_3 = \omega_M$ of 11 possible cases $\Omega_3^2 = \omega_M^2 = \omega_s^2$, $M = s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ contain one term and in the form $\mathrm{H}(\omega_M t) = \left((\omega_M t)\Delta_k\left(\omega_M^2\right)/2\omega_M^2\Delta^{(N-1)}\left(\omega_M^2\right)\right)\sin\omega_M t$ linearly depending on time and increasing with time. This is illustrated in Figure 4.

In Figure 5 eleven amplitude-frequency resonant curves of a non-homogeneous chain system at nano level with 11 d.f. for forced vibrations under the action of one frequency: external excitation applied to the third mass particle in chain or electric voltage applied to the third electric sub-circuit to the coil inductor and capacitator or mechanical couple applied to the third pendulum in a multi-pendulum chain system are presented.

From 11 amplitude-frequency curves, presented in Figure 5, it can be seen that dynamical absorption appears in the forced vibrations of:

- the first mass particle at five circular frequencies of external one frequency excitation;
- the second mass particle at five circular frequencies of external one frequency excitation;
- the third mass particle at three circular frequencies of external one frequency excitation;
- the fourth mass particle at five circular frequencies of external one frequency excitation;
- the fifth mass particle at three circular frequencies of external one frequency excitation;
- the sixth mass particle at two circular frequencies of external one frequency excitation;
- the seventh mass particle at four circular frequencies of external one frequency excitation;
- the eighth mass particle at four circular frequencies of external one frequency excitation;
- the ninth mass particle at zero circular frequencies of external one frequency excitation;
- the tenth mass particle at one circular frequency of external one frequency excitation;
- the eleventh mass particle at one circular frequency of external one frequency excitation.

The appearance of one or more regimes of dynamic absorptions at the resonant frequencies is possible, so it is also possible that some of the mass particles in the chain are in the regime of resonant oscillations, while other mass particles are in the regime of dynamic absorption. This is evident for example in Figure 5, which presents amplitude-frequency curves of each mass particle in the chain, while the amplitude-frequency curves possess roots at the same point as the characteristic frequency graph.

6. Conclusions

In conclusion, we believe that Petrović's theory of mathematical phenomenology elements, phenomenological mapping and mathematical analogy is a very useful tool for the integration of knowledge obtained in various areas of science on the basis of phenomenological mappings, analogous models of the dynamics of systems of disparate natures (mechanical, electrical, biomechanical, physico-chemical, socio-economical) and a transfer of knowledge and obtained research results from one area of science to another.

We can further consider transversal vibrations of an 11 deformable beam hybrid system on a discrete continuum layer with linear elastic and translator and rotator inertia properties described by 11 coupled partial differential equations along the beam's transversal displacements $w_k(x,t)$, $k = 1, 2, 3, \ldots, 11$ (see Rašković, 1952; Hedrih (Stevanović), 2006a, 2008a). The next two examples for consideration are the transversal vibrations of an 11 deformable plate hybrid system as well as the transversal vibrations of an 11 deformable membrane hybrid system, on a discrete continuum layer with ideal linear elastic and translator and rotator inertia properties. Comparing these three listed systems of partial differential equations, each describing transversal vibrations of a hybrid system consisting each of 11 deformable bodies of the same type (beams, or plates or membranes) coupled by discrete continuum layers with translator and rotator inertia properties, which vibrate on foundations with an ideal elastic layer with translator and rotator inertia properties, a similar form and a mathematical analogy can be identified. Also, for solving these coupled partial differential equations, a similar series can be used along eigen amplitude functions and eigen 11 frequency time functions, $T_{k(n)}(t)$ for an 11 beam system and $T_{k(nm)}(t)$, $k = 1, 2, 3, \ldots, 11$, $n, m = 1, 2, 3, 4 \ldots \infty$ for an 11 plate system, as well as for 11 membrane systems. Then phenomenological mapping and mathematical analogy between eigen time functions corresponding to eigen amplitude functions is obvious. In the case of distributed external one frequency excitation applied along the third body (along the beam, the plate surface or the membrane surface), in eigen amplitude mode, it is possible to transform a problem's solution into a solution system of 11 ordinary differential equations along unknown eigen time functions $T_{k(nm)}(t)$, $k = 1, 2, 3, \ldots, 11$, $n, m = 1, 2, 3, 4 \ldots \infty$, for each of eigen amplitude functions. These ordinary differential equations are of the same type as the corresponding homogenous chain system with 11 d.f.. Eigen time functions $T_{k(nm)}(t)$, $k = 1, 2, 3, \ldots, 11$, $n, m = 1, 2, 3, 4 \ldots \infty$ for each of eigen amplitude functions are in analogy with homogeneous chain mass particle displacements.

Using phenomenological mapping, data obtained for free and forced vibrations can be used for qualitative explanation of characteristic time functions of transversal vibration of multi-deformable body hybrid systems.

It is then possible to discuss a set of eigen circular frequencies of free transversal vibrations, as well as the appearance of resonance regimes and dynamical absorbtion regimes of time function corresponding to a deformable body (beam, plate or membrane) in multi body system forced vibrations in corresponding eigen amplitude form.

Briefly, the novelty of this work is in studying chain dynamics in systems with multiple d.f. (11) using phenomenological mapping.

The application of this method covers eigen and forced chain dynamics in systems with 11 d.f.: chains, beams, plates, twisted chains, pendulums, electrical chains. For all these systems we determined a set of eigen circular frequencies, demonstrating the phenomenon of dynamical absorption and resonance. Phenomenological mapping enables us to make a transfer of conclusions from the analysis of dynamical properties and phenomena in one chain system to dynamical properties in another chain system with the same d.f..

The results of this study are applicable to biological chain oscillators, such as a DNA double helix, (Hedrih (Stevanović) and Hedrih, 2010; Hedrih, 2011, 2012). Also, we can point out that discrete continuum method (Hedrih (Stevanović), 2002, 2006b, 2008b, 2009) is based on coupled chains as a model abstraction of real continuum in the results of discretizations of structures. The results presented here can be applied to models obtained by a discrete continuum method to solve defined tasks of the dynamics of systems of different kinds.

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References

Ekwaro-Osire S and Desen IC (2001) Experimental study on an impact vibration absorber. *Journal of Vibration and Control* 7(4): 475–493.

Filipovic D and Schroder D (1999) Vibration absorption with linear active resonators: continuous and discrete time design and analysis. *Journal of Vibration and Control* 5(5): 685–708.

Frazer RW (1961) Applications of conformal mapping to the phenomenological representation of scattering amplitudes. *Physical Review* 123(6): 2180–2182.

Freudenthal H (1986) *Didactical Phenomenology of Mathematical Structures*. Berlin: Springer.

Gleick J (1987) *Chaos: Making a New Science*. New York: Vintage.

Hedrih (Stevanović) K (1991) Analogy between models of stress state, strain state and state of the body mass inertia moments. *Facta Universitatis, Series Mechanics, Automatic Control and Robotics* 1(1): 105–120.

Hedrih (Stevanovic) K (2002) Discrete continuum method. In *Proceedings of the Recent Advances in Analytical Dynamics Control, Stability and Differential Geometry Symposium* (ed Djordjevic), Belgrade, Serbia, 25–26 May, pp. 30–57. Belgrade: Mathematical institute SANU.

Hedrih (Stevanović) K (2006a) Transversal vibrations of double-plate systems. *Acta Mechanica Sinica* 22: 487–501.

Hedrih (Stevanović) K (2006b) Modes of the homogeneous chain dynamics. *Signal Processing* 86: 2678–2702.

Hedrih (Stevanović) K (2008a) Dynamics of coupled systems. *Nonlinear Analysis: Hybrid Systems* 2(2): 310–334.

Hedrih (Stevanović) Katica (2008b) Vibration modes of an axially moving double belt system with creep layer. *Journal of Vibration and Control* 14(9–10): 1333–1347.

Hedrih (Stevanović) K (2009) Considering transfer of signals through hybrid fractional order homogeneous structure, keynote lecture. In *Proceedings of Selected AAS (Applied Automatic Systems) 2009 Papers* (ed G. Dimirovski), Ohird, Makedonija, 26–29 September, pp. 19–24. Skopje –Istanbul: National Library of R. Makedonia and ETAI Society.

Hedrih (Stevanović) K and Hedrih A (2010) Eigen modes of the double DNA chain helix vibrations. *Journal of Theoretical and Applied Mechanics* 1(48): 219–231.

Hedrih A (2011) Modeling oscillations of Zona Pellucida before and after fertilization. ENOC Young Scientist Prize Paper. *EUROMECH Newsletter* 40: 6–14.

Hedrih A (2012) Frequency analysis of knot mass particles in oscillatory spherical net model of mouse zona pellucida. Lecture Session, Short Paper. In: *Abstract book of 23rd International Congress of Theoretical and Applied Mechanics, (IUTAM ICTAM Beijing 2012)*, Beijing, China, 19–24 August, SM01-049, pp. 209. Beijing: UITAM and The Chinese Society of Theoretical and Applied Mechanics.

Kelebanov RI and Maldacena MJ (2009) Solving quantum field theories via curved spacetimes. *Physics Today* 68: 28–33.

Najar F, Nayfeh AH, Abdel-Rahman EM, et al. (2010) Nonlinear analysis of MEMS electrostatic microactuators:

primary and secondary resonances of the first mode. *Journal of Vibration and Control* 16(9): 1321–1349.
Penrose R (1989) *The Emperor's New Mind: Concerning Computers, Minds and The Laws of Physics*. Oxford: Oxford University Press.
Pettifor DG (1986) The structures of binary compounds. I. Phenomenological structure maps. *Journal of Physics. C: Solid State Physics* 19: 285.
Petrović M (1911) *Elementi matematičke fenomenologije* [Elements of mathematical phenomenology]. Beograd: Srpska kraljevska akademijahttp://elibrary.matf.bg.ac.rs/handle/123456789/476?locale-attribute = sr (accessed 2 August 2012).
Petrović M (1933) *Fenomenološko preslikavanje* [Phenomenological mapping]. Beograd: Srpska kraljevska akademija.
Pfeiffer F, Fritz P and Srnik J (1997) Nonlinear vibrations of chains. *Journal of Vibration and Control* 3(4): 397–410.
Rašković D (1952) *Teorija oscilacija* [Theory of oscillations]. Naučna knjiga.
Rašković D (1972) *Mehanika - Dinamika* [Dynamics]. Naučna knjiga.
Rašković D (1974) *Analitička mehanika* [Analytical mechanics]. Mašinski fakultet KragujevacJ90H.
Walsh R (1995) Phenomenological mapping: a method for describing and comparing states of consciousness. *Journal of Transpersonal Psychology* 27: 125–156.