A note on the Drazin inverse of a modified matrix

Abstract

In this paper the expression for the Drazin inverse of a modified matrix is considered and some interesting results are established. This contributes to certain recent results obtained by Y.Wei[9].
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1 Introduction

Let $C^{n\times m}$ denote the set of all complex $n \times m$ matrices. For $A \in C^{n\times m}$, the set of inner inverses are given by:

$$A\{1\} = \{X : AXA = A\}. \tag{1}$$

Let us recall that the Drazin inverse of $A \in C^{n\times n}$ [3] is the matrix $A^D \in C^{n\times n}$ which satisfies

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

for some nonnegative integer k. The least such k is the index of A, denoted by $ind(A)$. Some interesting properties of Drazin inverse, among other papers, are investigated in [8], [10], [4].

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In this paper we consider a matrix $A \in C^{(m+p)\times(n+q)}$ partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2}$$

where $A \in C^{m\times n}$ and $D \in C^{p\times q}$.

The motivation for this research is the paper of Y.Wei [9] in which he derives various expressions for the Drazin inverse of a modified matrix.

It is well-known that the generalized Schur complement of D in M is defined as:

$$S(M) = A - BD^{-}C, \tag{3}$$

where $D^{-} \in D\{1\}$.

If we replace $D^{-} \in D\{1\}$ by the Drazin inverse of D in (3), we obtain the Drazin-Schur complement of D in M, which we denote by

$$S_D(M) = A - BD^DC.$$

The Drazin-Schur complement of A in M, is denoted by

$$Z_D(M) = D - CA^DB.$$

For interesting results concerning Schur complements see [1], [2], [6], [7].

In this paper we derive some expressions for the Drazin inverse of Drazin-Schur complement for the matrix M given by (2). As a corollary, we obtain the results of Wei [9].

2 Results

For an arbitrary matrix A we denote by $E_A = I - AA^D$. Let

$$K = A^DB, \quad H = CA^D, \quad G = HK.$$

We use S and Z instead of $S_D(M)$ and $Z_D(M)$, respectively.

When the partitioned matrix M and the submatrix D are both nonsingular, then the Schur complement of D in M is also nonsingular. When M, A and D are all three nonsingular , then

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^DB)^{-1}CA^{-1}$$

which was observed by Duncan [5]. We have the analogous result concerning the Drazin inverse and the Drazin-Schur complement.

Theorem 2.1 *Suppose that* $E_AB = 0$, $CE_A = 0$, $BE_DZ^DC = 0$, $BD^DE_ZC = 0$, $BZ^DE_DC = 0$, $BE_ZD^DC = 0$. *Then*

$$S^D = A^D + A^DBZ^DCA^D.$$

Proof. Let $X = A^D + A^DBZ^DCA^D$. Then

$$\begin{aligned} SX &= (A - BD^DC)(A^D + A^DBZ^DCA^D) \\ &= AA^D + AA^DBZ^DCA^D - BD^DCA^D - BD^DCA^DBZ^DCA^D \\ &= AA^D + BZ^DCA^D - BD^DCA^D - BD^D(D - Z)Z^DCA^D \\ &= AA^D + BE_DZ^DCA^D - BD^DE_ZCA^D \\ &= AA^D. \end{aligned}$$

Similarly, $XS = A^DA$ i.e. $XS = SX$. Further

$$\begin{aligned} XSX &= A^DA(A^D + A^DBZ^DCA^D) \\ &= A^D + A^DBZ^DCA^D \\ &= X. \end{aligned}$$

By induction, it follows that

$$(A - BD^DC)^{m+1}X = (A - BD^DC)^m + (A^{m+1}A^D - A^m).$$

Hence, $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$ holds for $m \geq index(A)$. $\square$

In the case when $D = I$, it follows that $E_D = 0$. Hence, we obtain Theorem 2.1 from [9] as a corollary of our Theorem 2.1.

Corollary 2.1 *If* $E_AB = 0$, $CE_A = 0$, $BE_ZC = 0$*, then*

$$(A - BC)^D = A^D + A^DBZ^DCA^D,$$

where $Z = I - CA^DB$.

If Z is invertible, then from Theorem 2.1 we get the following result:

Corollary 2.2 *Suppose that Z is nonsingular and $E_A B = 0$, $CE_A = 0$, $BE_D Z^{-1} C = 0$, $BZ^{-1} E_D C = 0$. Then*

$$S^D = A^D + A^D BZ^{-1} CA^D.$$

In the case when $B = I$ we have the following corollary:

Corollary 2.3 *Suppose that $CE_A = 0$, $E_A D^D = 0$ and $\|A^D\| \cdot \|D^D C\| \leq 1$. Then*

$$(A - D^D C)^D = (I - A^D D^D C)^{-1} A^D = A^D (I - D^D CA^D)^{-1}$$

and

$$(A - D^D C)^D - A^D = (A - D^D C)^D D^D CA^D = A^D D^D C(A - D^D C)^D,$$

with

$$\frac{\|(A - D^D C)^D - A^D\|}{\|A^D\|} \leq \frac{k_D(A)\|D^D C\|/\|A\|}{1 - k_D(A)\|D^D C\|/\|A\|},$$

where $k_D(A) = \|A\|\|A^D\|$ is the condition number with respect to the Drazin inverse.

Proof. For the proof of this corollary see Theorem 3.2 and Corollary 3.2 from [10].

Theorem 2.2 *Let $Z = 0$, $E_A B = 0$, $CE_A = 0$, $BE_D G^D C = 0$, $BD^D E_G C = 0$, $BG^D E_D C = 0$ and $BE_G D^D C = 0$. Then*

$$\begin{aligned} S^D &= (I - KG^D H)A^D (I - KG^D H) \\ &= (I - KH(KH)^D)A^D (I - KH(KH)^D). \end{aligned}$$

Proof. Denote by $X = (I - KG^D H)A^D (I - KG^D H)$. We obtain that

$$\begin{aligned} SX &= (A - BD^D C)(I - KG^D H)A^D (I - KG^D H) \\ &= (A - BD^D C - BG^D CA^D + BD^D DG^D CA^D) \\ &\quad \times (A^D - (A^D)^2 BG^D CA^D) \\ &= (A - BD^D C)(A^D - (A^D)^2 BG^D CA^D) \end{aligned}$$

$$\begin{aligned}
&= AA^D - BD^DCA^D - A(A^D)^2BG^DCA^D \\
&\quad +BD^DC(A^D)^2BG^DCA^D \\
&= AA^D - BD^DCA^D - A^DBG^DCA^D + BD^DGG^DCA^D \\
&= AA^D - A^DBG^DCA^D \\
&= AA^D - KG^DH
\end{aligned}$$

and $XS = AA^D - KG^DH$, i.e. $XS = SX$. Also,

$$\begin{aligned}
XSX &= (AA^D - KG^DH)(I - KG^DH)A^D(I - KG^DH) \\
&= (AA^D - KG^DH - AA^DKG^DH + KG^DHKG^DH) \\
&\quad \times(A^D - A^DKG^DH) \\
&= (AA^D - KG^DH)(A^D - A^DKG^DH) \\
&= (I - KG^DH)AA^DA^D(I - KG^DH) \\
&= X.
\end{aligned}$$

We prove that $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$ by induction. □

If $D = I$, then we obtain the Theorem 2.2 of [9]:

Corollary 2.4 *Suppose that* $Z = 0$, $E_AB = 0$, $CE_A = 0$ *and* $BE_GC = 0$. *Then*

$$\begin{aligned}
(A - BC)^D &= (I - KG^DH)A^D(I - KG^DH) \\
&= (I - KH(KH)^D)A^D(I - KH(KH)^D).
\end{aligned}$$

Theorem 2.3 *Let* $ind(Z) = 1$ *and* $E_AB = 0$, $CE_A = 0$, $BE_D = 0$, $E_DC = 0$, $ZZ^\#G = GZZ^\#$, $BD = DB$, $CD = DC$, $BE_GC = 0$. *Then*

$$S^D = (I - KE_ZG^DH)A^D(I - KE_ZG^DH) + KZ^\#H. \tag{4}$$

Proof. Denote by X the right side of (4). We have that

$$\begin{aligned}
SX &= (A - BD^DC - BE_ZG^DH + BD^DCA^DBE_ZG^DH) \\
&\quad \times A^D(I - KE_ZG^DH) + AKZ^\#H - BD^DCA^DBZ^\#H \\
&= (A - BD^DC - BE_ZG^DH + BD^D(D - Z)E_ZG^DH) \\
&\quad \times A^D(I - KE_ZG^DH) + BZ^\#H - BD^D(D - Z)Z^\#H
\end{aligned}$$

$$
\begin{aligned}
&= (A - BD^DC)A^D(I - KE_ZG^DH) + BD^DZZ^{\#}H \\
&= AA^D - KE_ZG^DH - BD^DCA^D + BD^DGG^DE_ZH \\
&\quad + BD^DZZ^{\#}H \\
&= AA^D - KE_ZG^DH - BD^DE_GCA^D + BD^DE_GZZ^{\#}CA^D \\
&= AA^D - KE_ZG^DH
\end{aligned}
$$

and

$$
\begin{aligned}
XS &= (I - KE_ZG^DH)A^D(A - BD^DC - KE_ZG^DC \\
&\quad + KE_ZG^DCA^DBD^DC) + KZ^{\#}C - KZ^{\#}CA^DBD^DC \\
&= (I - KE_ZG^DH)A^D(A - BD^DC - KE_ZG^DC \\
&\quad + KE_ZG^D(D - Z)D^DC) + KZ^{\#}C - KZ^{\#}(D - Z)D^DC \\
&= (I - KE_ZG^DH)A^D(A - BD^DC) + KZ^{\#}ZD^DC \\
&= A^DA - A^DBD^DC - KE_ZG^DH + KE_ZG^DGD^DC \\
&\quad + KZ^{\#}ZD^DC \\
&= A^DA - KG^DE_ZH - A^DBE_GD^DC + KZ^{\#}ZE_GD^DC \\
&= A^DA - KG^DE_ZH.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
XSX &= (A^DA - KG^DE_ZH)(I - KE_ZG^DH)A^D \times \\
&\quad (I - KE_ZG^DH) + (A^DA - KG^DE_ZH)KZ^{\#}H \\
&= (A^DA - KG^DE_ZH)A^D(I - KE_ZG^DH) + KZ^{\#}H \\
&= X.
\end{aligned}
$$

By induction, it follows that

$$(A - BD^DC)^{m+1}X = (A - BD^DC)^m + (A^{m+1}A^D - A^m).$$

Hence, $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$, for $m \geq ind(A)$.

Obviously, for $D = I$ we have the following result:

Corollary 2.5 *Let* $E_AB = 0,\ CE_A = 0,\ BE_GC = 0,\ G^DE_Z = E_ZG^D$ *and* $index(Z) = 1$. *Then*

$$(A - BC)^D = (I - KE_ZG^DH)A^D(I - KE_ZG^DH) + KZ^{\#}H.$$

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