Representations for the Drazin inverse of block matrix*

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Abstract

In this paper we offer new representations for Drazin inverse of block matrix, which recover some representations from current literature on this subject.

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1 Introduction

Let A be a square complex matrix. By $\text{rank}(A)$ we denote the rank of a matrix A. The index of a matrix A, denoted by $\text{ind}(A)$, is the smallest nonnegative integer k such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$. For every matrix $A \in \mathbb{C}^{n\times n}$, such that $\text{ind}(A) = k$, there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies following relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

Matrix A^d is called the Drazin inverse of matrix A (see [1]). In the case $\text{ind}(A) = 1$, the Drazin inverse of A is called the group inverse of A, denoted by $A^{\#}$ or A^g. The case $\text{ind}(A) = 0$ is valid if and only if A is nonsingular, so in that case A^d reduces to A^{-1}. Throughout this paper we suppose that $A^0 = I$, where I is identity matrix, and $\sum_{i=1}^{k-j} * = 0$, for $k \leq j$.

The theory of Drazin inverse of a square matrix has numerous applications, such as in singular differential equations and singular difference

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equations, Markov chains and iterative methods (see [2, 4, 5, 6, 8, 9]). An application of the Drazin inverse of a 2×2 block matrix can be found in [2, 3, 7].
In 1979 Campbell and Meyer[4] posed the problem of finding an explicit representation for the Drazin inverse of 2×2 complex matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

in terms of its blocks, where A and D are square matrices, not necessarily of the same size. Until now, there has been no formula for M^d without any side conditions for blocks of matrix M. However, many papers studied special cases of this open problem and offered a formula for M^d under some specific conditions for blocks of M. Here we list some of them:

(i) $B = 0$ (or $C = 0$) (see [10, 11]);

(ii) $BC = 0$, $BD = 0$ and $DC = 0$ (see [6]);

(iii) $BC = 0$, $DC = 0$ (or $BD = 0$) and D is nilpotent (see [7]);

(iv) $BC = 0$ and $DC = 0$ (see [12]);

(v) $CB = 0$ and $AB = 0$ (or $CA = 0$) (see [12, 13]);

(vi) $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$ (see [14]);

(vii) $ABC = 0$, $CBC = 0$, $ABD = 0$ and $CBD = 0$ (see [14]);

(viii) $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$ (see [15]);

(ix) $BCA = 0$, $DCA = 0$, $CBC = 0$, and $CBD = 0$ (see [15]);

(x) $BCA = 0$, $BD = 0$ and $DC = 0$ (or BC is nilpotent) (see [16]);

(xi) $BCA = 0$, $DC = 0$ and D is nilpotent (see [16]);

(xii) $ABC = 0$, $DC = 0$ and $BD = 0$ (or BC is nilpotent, or D is nilpotent) (see [17]);

(xiii) $BCA = 0$ and $BD = 0$ (see [18]);

(xiv) $ABC = 0$ and $DC = 0$ (or $BD = 0$) (see [18, 19]).

In this paper we derive representations for M^d which recover representations from previous list.

2 Key lemmas

In order to prove our main results, we first state some lemmas.

Lemma 2.1 [14] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$. *If* $PQP = 0$ *and* $PQ^2 = 0$ *then*

$$(P+Q)^d = Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2 P^d\right) PQ,$$

where

$$Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1},\ Y_2 = \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi. \tag{2.1}$$

Lemma 2.2 [14] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$. *If* $QPQ = 0$ *and* $P^2Q = 0$ *then*

$$(P+Q)^d = Y_1 + Y_2 + PQ\left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2 P^d\right),$$

where Y_1 *and* Y_2 *are defined by* (2.1).

Lemma 2.3 [20] *Let* $M \in \mathbb{C}^{n\times n}$ *be such that* $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $B \in \mathbb{C}^{p\times(n-p)}$, $C \in \mathbb{C}^{(n-p)\times p}$. *Then*

$$M^d = \begin{bmatrix} 0 & B(CB)^d \\ (CB)^d C & 0 \end{bmatrix}.$$

Deng and Wei [21] gave representations for the Drazin inverse of upper anti-triangular block matrix under some specific conditions. Here we state these results and some additional facts, which we will be useful to prove our results. Consider the block matrix of a form (1.1), where $D = 0$:

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.2}$$

Lemma 2.4 [21] *Let* $M \in \mathbb{C}^{n\times n}$ *be matrix of a form* (2.2). *If* $ABC = 0$, *then*

$$M^d = \begin{bmatrix} \Phi A & \Phi B \\ C\Phi & C\Phi^2 AB \end{bmatrix},$$

where

$$\Phi = (A^2 + BC)^d = \sum_{i=0}^{t_1-1} (BC)^\pi (BC)^i (A^d)^{2i+2} + \sum_{i=0}^{\nu_1-1} ((BC)^d)^{i+1} A^{2i} A^\pi \tag{2.3}$$

and $t_1 = \operatorname{ind}(BC)$, $\nu_1 = \operatorname{ind}(A^2)$.

Remark 1 *Let M be matrix of a form* (2.2)*. If conditions of* Lemma 2.4 *are satisfied, we have that:*

$$M^{2k+1} = \begin{bmatrix} (A^2+BC)^k A & (A^2+BC)^k B \\ C(A^2+BC)^k & C(A^2+BC)^{k-1}AB \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} (A^2+BC)^k & (A^2+BC)^{k-1}AB \\ C(A^2+BC)^{k-1}A & C(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Notice that $(A^2+BC)^k = \sum_{j=0}^{k}(BC)^{k-j}A^{2j}$*, for* $k \geq 0$*. Also,* $(A^2+BC)^\pi = A^\pi - BC\Phi = (BC)^\pi - \Phi A^2$*. We can check that*

$$\Phi^k = \sum_{i=0}^{t_1-1}(BC)^\pi (BC)^i (A^d)^{2i+2k} + \sum_{i=0}^{\nu_1-1}((BC)^d)^{i+k}A^{2i}A^\pi - \sum_{i=1}^{k-1}((BC)^d)^{k-i}(A^d)^{2i},$$

for $k \geq 1$*. Therefore we have*

$$(M^d)^{2k+1} = \begin{bmatrix} \Phi^{k+1}A & \Phi^{k+1}B \\ C\Phi^{k+1} & C\Phi^{k+2}AB \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} \Phi^k & \Phi^{k+1}AB \\ C\Phi^{k+1}A & C(\Phi^{k+1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Lemma 2.5 [21] *Let $M \in \mathbb{C}^{n\times n}$ be as in* (2.2)*. If $BCA = 0$, then*

$$M^d = \begin{bmatrix} A\Omega & \Omega B \\ C\Omega & CA\Omega^2 B \end{bmatrix},$$

where

$$\Omega = (A^2+BC)^d = \sum_{i=0}^{t_1-1}(A^d)^{2i+2}(BC)^i(BC)^\pi + \sum_{i=0}^{\nu_1-1}A^\pi A^{2i}((BC)^d)^{i+1} \quad (2.4)$$

and $t_1 = \mathrm{ind}(BC)$, $\nu_1 = \mathrm{ind}(A^2)$.

Remark 2 *Let M be matrix of a form* (2.2). *If conditions of* Lemma 2.5 *hold, we have that:*

$$M^{2k+1} = \begin{bmatrix} A(A^2+BC)^k & (A^2+BC)^k B \\ C(A^2+BC)^k & CA(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} (A^2+BC)^k & A(A^2+BC)^{k-1}B \\ CA(A^2+BC)^{k-1} & C(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Clearly, $(A^2+BC)^k = \sum_{j=0}^{k} A^{2j}(BC)^{k-j}$, *for* $k \geq 0$. *Also* $(A^2+BC)^\pi = A^\pi - \Omega BC = (BC)^\pi - A^2\Omega$. *Furthermore, we have that*

$$\Omega^k = \sum_{i=0}^{t_1-1}(A^d)^{2i+2k}(BC)^i(BC)^\pi + \sum_{i=0}^{\nu_1-1} A^\pi A^{2i}((BC)^d)^{i+k} - \sum_{i=1}^{k-1}(A^d)^{2i}((BC)^d)^{k-i},$$

for $k \geq 1$. *Hence we get that*

$$(M^d)^{2k+1} = \begin{bmatrix} A\Omega^{k+1} & \Omega^{k+1}B \\ C\Omega^{k+1} & CA\Omega^{k+2}B \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} \Omega^k & A\Omega^{k+1}B \\ CA\Omega^{k+1} & C\Omega^{k+1}B \end{bmatrix}, \text{ for } k \geq 1.$$

In following two lemmas we present two new representations for Drazin inverse of lower anti-triangular block matrix. Consider the block matrix of a form (1.1) such that $A = 0$:

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}. \tag{2.5}$$

Lemma 2.6 *Let $M \in \mathbb{C}^{n\times n}$ be matrix of a form* (2.5). *If $DCB = 0$, then*

$$M^d = \begin{bmatrix} B\Psi^2 DC & B\Psi \\ \Psi C & \Psi D \end{bmatrix},$$

where

$$\Psi = (D^2+CB)^d = \sum_{i=0}^{t_2-1}(CB)^\pi(CB)^i(D^d)^{2i+2} + \sum_{i=0}^{\nu_2-1}((CB)^d)^{i+1}D^{2i}D^\pi \tag{2.6}$$

and $t_2 = \operatorname{ind}(CB)$, $\nu_2 = \operatorname{ind}(D^2)$.

Proof. First, notice that from $DCB = 0$ we have that matrices D^2 and CB satisfy the conditions of Lemma 2.1. Hence we get

$$(D^2 + CB)^d = \sum_{i=0}^{t_2-1} (CB)^\pi (CB)^i (D^d)^{2i+2} + \sum_{i=0}^{\nu_2-1} ((CB)^d)^{i+1} D^{2i} D^\pi.$$

Consider the splitting of matrix M

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} := P + Q.$$

Since $DCB = 0$ we have that $PQ^2 = 0$. Also, we have $PQP = 0$. Therefore matrices P and Q satisfy the conditions of Lemma 2.1 and

$$(P+Q)^d = Y_1+Y_2+\left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2 P^d\right) PQ, \tag{2.7}$$

where Y_1, Y_2 are as in (2.1). Clearly,

$$Q^{2k} = \begin{bmatrix} (BC)^k & 0 \\ 0 & (CB)^k \end{bmatrix}, \; Q^{2k+1} = \begin{bmatrix} 0 & B(CB)^k \\ (CB)^k C & 0 \end{bmatrix}, \text{for } k \geq 0.$$

Furthermore, by Lemma 2.3 we have

$$(Q^d)^{2k} = \begin{bmatrix} B((CB)^d)^{k+1} & 0 \\ 0 & ((CB)^d)^k \end{bmatrix}, \text{for } k \geq 1,$$

$$(Q^d)^{2k+1} = \begin{bmatrix} 0 & B((CB)^d)^{k+1} \\ ((CB)^d)^{k+1} C & 0 \end{bmatrix}, \text{for } k \geq 0.$$

After computing, we get

$$Y_1 = \begin{bmatrix} 0 & B\displaystyle\sum_{i=0}^{t_2-1} (CB)^\pi (CB)^i (D^d)^{2i+2} \\ 0 & \displaystyle\sum_{i=0}^{t_2-1} (CB)^\pi (CB)^i (D^d)^{2i+1} \end{bmatrix}, \tag{2.8}$$

$$Y_2 = \begin{bmatrix} 0 & B\displaystyle\sum_{i=0}^{\nu_2-1} ((CB)^d)^{i+1} D^{2i} D^\pi \\ (CB)^d C & \displaystyle\sum_{i=0}^{\nu_2-1} ((CB)^d)^{i+1} D^{2i+1} D^\pi \end{bmatrix}. \tag{2.9}$$

After substituting (2.8) and (2.9) into (2.7) we get that the statement of the lemma is valid. □

Remark 3 *Let M be matrix of a form (2.5) such that $DCB = 0$. Then*

$$M^{2k+1} = \begin{bmatrix} B(D^2+CB)^{k-1}DC & B(D^2+CB)^k \\ (D^2+CB)^kC & (D^2+CB)^kD \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} B(D^2+CB)^{k-1}C & B(D^2+CB)^{k-1}D \\ (D^2+CB)^{k-1}DC & (D^2+CB)^k \end{bmatrix}, \text{ for } k \geq 1.$$

It can be checked easily that $(D^2+CB)^k = \sum_{j=0}^{k}(CB)^{k-j}D^{2j}$, *for* $k \geq 0$, *and* $(D^2+CB)^\pi = D^\pi - CB\Psi = (CB)^\pi - \Psi D^2$. *Also, we have that*

$$\Psi^k = \sum_{i=0}^{t_2-1}(CB)^\pi(CB)^i(D^d)^{2i+2k} + \sum_{i=0}^{\nu_2-1}((CB)^d)^{i+k}D^{2i}D^\pi - \sum_{i=1}^{k-1}((CB)^d)^{k-i}(D^d)^{2i},$$

for $k \geq 1$. *Therefore we get*

$$(M^d)^{2k+1} = \begin{bmatrix} B\Psi^{k+2}DC & B\Psi^{k+1} \\ \Psi^{k+1}C & \Psi^{k+1}D \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} B\Psi^{k+1}C & B\Psi^{k+1}D \\ \Psi^{k+1}DC & \Psi^k \end{bmatrix}, \text{ for } k \geq 1.$$

Using the similar method as in the proof of Lemma 2.6 we can get the following result.

Lemma 2.7 *Let $M \in \mathbb{C}^{n\times n}$ be as in (2.5). If $CBD = 0$, then*

$$M^d = \begin{bmatrix} BD\Gamma^2C & B\Gamma \\ \Gamma C & D\Gamma \end{bmatrix},$$

where

$$\Gamma = \sum_{i=0}^{t_2-1}(D^d)^{2i+2}(CB)^i(CB)^\pi + \sum_{i=0}^{\nu_2-1}D^\pi D^{2i}((CB)^d)^{i+1} \tag{2.10}$$

and $t_2 = \mathrm{ind}(CB)$, $\nu_2 = \mathrm{ind}(D^2)$.

Proof. Since $CBD = 0$, using Lemma 2.1 we get (2.10). Now, if we split matrix M as

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} := P + Q,$$

we have that $QPQ = 0$ and $P^2Q = 0$. Hence, the conditions of Lemma 2.2 are satisfied. After applying Lemma 2.2 and Lemma 2.3 we complete the proof.□

Remark 4 *Let M be as in (2.5) and let $CBD = 0$. Then*

$$M^{2k+1} = \begin{bmatrix} BD(D^2+CB)^{k-1}C & B(D^2+CB)^k \\ (D^2+CB)^kC & D(D^2+CB)^k \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} B(D^2+CB)^{k-1}C & BD(D^2+CB)^{k-1} \\ (D^2+CB)^{k-1}C & (D^2+CB)^k \end{bmatrix}, \text{ for } k \geq 1.$$

Clearly $(D^2 + CB)^k = \sum_{j=0}^{k} D^{2j}(CB)^{k-j}$, *for* $k \geq 0$, *and* $(D^2 + CB)^\pi = D^\pi - \Gamma CB = (CB)^\pi - D^2\Gamma$. *In addition, we can get that*

$$\Gamma^k = \sum_{i=0}^{t_2-1}(D^d)^{2i+2k}(CB)^i(CB)^\pi + \sum_{i=0}^{\nu_2-1} D^\pi D^{2i}((CB)^d)^{i+k} - \sum_{i=1}^{k-1}(D^d)^{2i}((CB)^d)^{k-i},$$

for $k \geq 1$. *Also, we can get that*

$$(M^d)^{2k+1} = \begin{bmatrix} BD\Gamma^{k+2}C & B\Gamma^{k+1} \\ \Gamma^{k+1}C & D\Gamma^{k+1} \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} B\Gamma^{k+1}C & BD\Gamma^{k+1} \\ D\Gamma^{k+1}C & \Gamma^k \end{bmatrix}, \text{ for } k \geq 1.$$

3 Representations

Consider the block matrix M of a form (1.1). Djordjević and Stanimirović [6] gave explicit representation for M^d under conditions $BC = 0$, $BD = 0$ and $DC = 0$. This result was extended to a case $BC = 0$, $DC = 0$ (see [12]). As another generalization of these results, Yang and Liu [14] gave the representation for M^d under conditions $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$. In the next theorem we derive an explicit representation for M^d under conditions $BCA = 0$, $DCA = 0$ and $DCB = 0$. Therefore we can see that the condition $BCB = 0$ from [14] is superfluous.

Theorem 3.1 *Let M be matrix of a form* (1.1) *such that $BCA = 0$, $DCA = 0$ and $DCB = 0$. Then*

$$M^d = \begin{bmatrix} A^d + \Sigma_0 C & B\Psi + A\Sigma_0 \\ \begin{matrix} \Psi C + CA\Sigma_1 C + C(A^d)^2 \\ -CA^d(B\Psi^2 D + AB\Psi^2)C \end{matrix} & D^d + C\Sigma_0 \end{bmatrix},$$

where

$$\Sigma_k = \left(V_1\Psi^k + (A^d)^{2k}V_2\right)D + A\left(V_1\Psi^k + (A^d)^{2k}V_2\right), \textit{for } k = 0, 1, \quad (3.1)$$

$$V_1 = \sum_{i=0}^{\nu_1 - 1} A^\pi A^{2i} B\Psi^{i+2}, \quad (3.2)$$

$$V_2 = \sum_{i=0}^{\mu_1 - 1} (A^d)^{2i+4} B(D^2 + CB)^i D^\pi - \sum_{i=0}^{\mu_1} (A^d)^{2i+2} B(CB)^i \Psi, \quad (3.3)$$

$\nu_1 = \mathrm{ind}(A^2)$, $\mu_1 = \mathrm{ind}(D^2 + CB)$ *and Ψ is defined by* (2.6).

Proof. Consider the splitting of matrix M

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} := P + Q.$$

Since $BCA = 0$ and $DCA = 0$ we get $P^2Q = 0$ and $QPQ = 0$. Hence matrices P and Q satisfy the conditions of Lemma 2.2 and

$$(P+Q)^d = Y_1 + Y_2 + PQY_1(P^d)^2 + PQ^dY_2 - PQQ^d(P^d)^2 - PQ^dP^d, \quad (3.4)$$

where Y_1 and Y_2 are as in (2.1). By the assumption of the theorem $DCB = 0$ we have that matrix P satisfy the conditions of Lemma 2.6. After applying Lemma 2.6 and using Remark 3, we get

$$Y_1 = \begin{bmatrix} (V_1 D + AV_1)C & A^\pi B\Psi + A(V_1 D + AV_1) \\ \Psi C & \Psi D \end{bmatrix}, \quad (3.5)$$

$$Y_2 = \begin{bmatrix} A^d + (V_2 D + AV_2)C & B\Psi - A^\pi B\Psi + A(V_2 D + AV_2) \\ 0 & 0 \end{bmatrix}, \quad (3.6)$$

where V_1 and V_2 are defined by (3.2) and (3.3), respectively. After substituting (3.5) and (3.6) into (3.4) and computing all elements of (3.4) we obtain the result. □
As a direct corollary of the previous theorem we get the following result.

Corollary 3.1 *Let M be as in* (1.1)*. If $DCB = 0$ and $CA = 0$ then*

$$M^d = \begin{bmatrix} A^d + \Sigma_0 C & B\Psi + A\Sigma_0 \\ \Psi C & \Psi D \end{bmatrix},$$

where Σ_0 is defined by (3.1) *and Ψ is given in* (2.6)*.*

Notice that Corollary 3.1, therefore and Theorem 3.1 is also a generalization of representation for M^d under conditions $CB = 0$ and $CA = 0$ which is given in [13].

The next result is a corollary of Theorem 3.1. Also, we can get the following result using the splitting $M = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} := P + Q$ and applying Lemma 2.1 and Lemma 2.5.

Corollary 3.2 *Let M be matrix of a form* (1.1)*. If $BCA = 0$ and $DC = 0$ then*

$$M^d = \begin{bmatrix} A\Omega & \Omega B + RD \\ C\Omega & D^d + CR \end{bmatrix},$$

where

$$\begin{aligned} R &= (R_1 + R_2)D + A(R_1 + R_2), \\ R_1 &= \sum_{i=0}^{\mu_2 - 1} A^\pi (A^2 + BC)^i B (D^d)^{2i+4} - \sum_{i=0}^{\mu_2} \Omega (BC)^i B (D^d)^{2i+2}, \\ R_2 &= \sum_{i=0}^{\nu_2 - 1} \Omega^{i+2} B D^{2i} D^\pi, \end{aligned}$$

$\nu_2 = \mathrm{ind}(D^2)$, $\mu_2 = \mathrm{ind}(A^2 + BC)$ and Ω is defined by (2.4)*.*

We remark that Corollary 3.2, hence and Theorem 3.1 is also extension of results from [16], where beside conditions $BCA = 0$ and $DC = 0$ additional condition $BD = 0$ (or D is nilpotent) is required.

Castro–González et al. (see [16]) gave explicit representation for M^d under conditions $BCA = 0$, $BD = 0$ and BC is nilpotent (or $DC = 0$).

This result was extended to a case when $BCA = 0$ and $BD = 0$ (see [18]). The following theorem is extension of these results.

Theorem 3.2 *Let M be matrix of a form* (1.1) *such that $BCA = 0$, $ABD = 0$ and $CBD = 0$. Then*

$$M^d = \begin{bmatrix} A\Omega + B(F_1 + F_2) & \begin{array}{c}\Omega B + BD(F_1\Omega + (D^d)^2F_2)B \\ +B(D^d)^2 - BD^d(CA + DC)\Omega^2 B\end{array} \\ C\Omega + D(F_1 + F_2) & D^d + (F_1 + F_2)B \end{bmatrix}, \quad (3.7)$$

where

$$F_1 = \sum_{i=0}^{\nu_2 - 1} D^\pi D^{2i}(CA + DC)\Omega^{i+2},$$

$$F_2 = \sum_{i=0}^{\mu_2 - 1}(D^d)^{2i+4}(CA + DC)(A^2 + BC)^i(BC)^\pi - \sum_{i=0}^{\mu_2}(D^d)^{2i+2}(CA + DC)A^{2i}\Omega,$$

$\nu_2 = \mathrm{ind}(D^2)$, $\mu_2 = \mathrm{ind}(A^2 + BC)$ and Ω is defined by (2.4)*.*

Proof. If we split matrix M as

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} := P + Q.$$

we have that $QPQ = 0$ and $P^2Q = 0$. Hence, matrices P and Q satisfy the conditions of Lemma 2.2. Since $BCA = 0$, matrix P satisfies conditions of Lemma 2.5. Using the similar method as in the proof of Theorem 3.1, after applying Lemma 2.2, Lemma 2.5 and using Remark 2, we get that (3.7) holds. □

Notice that Theorem 3.2 is also generalization of representation from [15] where additional condition $BCB = 0$ is required.

In [15] a formula for M^d is given under conditions $BCA = 0$, $DCA = 0$, $CBD = 0$ and $CBC = 0$. In the next theorem we offer a representation for M^d under conditions $BCA = 0$, $DCA = 0$ and $CBD = 0$, without additional condition $CBC = 0$.

Theorem 3.3 *Let M be as in* (1.1)*. If $BCA = 0$, $DCA = 0$ and $CBD = 0$ then*

$$M^d = \begin{bmatrix} A^d + (G_1 + G_2)C & B\Gamma + A(G_1 + G_2) \\ \begin{array}{c}\Gamma C + CA(G_1\Gamma + (A^d)^2G_2)C \\ +C(A^d)^2 - CA^d(AB + BD)\Gamma^2 C\end{array} & D\Gamma + C(G_1 + G_2) \end{bmatrix},$$

where

$$G_1=\sum_{i=0}^{\nu_1-1}A^\pi A^{2i}(AB+BD)\Gamma^{i+2}, \tag{3.8}$$

$$G_2=\sum_{i=0}^{\mu_1-1}(A^d)^{2i+4}(AB+BD)(D^2+CB)^i(CB)^\pi-\sum_{i=0}^{\mu_1}(A^d)^{2i+2}(AB+BD)D^{2i}\Gamma, \tag{3.9}$$

$\nu_1=\text{ind}(A^2)$, $\mu_1=\text{ind}(D^2+CB)$ *and* Γ *is given in* (2.10).

Proof. Using the splitting of matrix M

$$M=\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}+\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}:=P+Q,$$

we get that conditions of Lemma 2.2 are satisfied. Also, we have that matrix P satisfies the conditions of Lemma 2.7. Using these lemmas and Remark 4, similarly as in the proof of Theorem 3.1, we get that the statement of the theorem is valid. □

Corollary 3.3 *Let* M *be matrix of a form* (1.1). *If* $CBD=0$ *and* $CA=0$, *then*

$$M^d=\begin{bmatrix} A^d+(G_1+G_2)C & B\Gamma+A(G_1+G_2) \\ \Gamma C & D\Gamma \end{bmatrix},$$

where Γ, G_1 *and* G_2 *are defined by* (2.10), (3.8) *and* (3.9) *respectively.*

We can see that Theorem 3.3 and Corollary 3.3 are also extensions of representation for M^d under conditions $CB=0$ and $CA=0$ (see [13]).

In [12] a representation for M^d is offered under conditions $AB=0$ and $CB=0$. This result was extended in [14], where a formula for M^d is given under conditions $ABC=0$, $ABD=0$, $CBD=0$ and $CBC=0$. In our following result we derive the representation for M^d under conditions $ABC=0$, $ABD=0$ and $CBD=0$, without additional condition $CBC=0$.

Theorem 3.4 *Let* M *be matrix of a form* (1.1). *If* $ABC=0$, $ABD=0$ *and* $CBD=0$. *Then*

$$M^d=\begin{bmatrix} A^d+B\Theta_0 & \begin{matrix} B\Gamma+B\Theta_1AB+(A^d)^2B \\ -B(\Gamma^2CA+D\Gamma^2C)A^dB \end{matrix} \\ \Gamma C+\Theta_0A & D^d+\Theta_0B \end{bmatrix}, \tag{3.10}$$

where

$$\Theta_k = \left(K_1(A^d)^{2k} + \Gamma^k K_2\right) A + D \left(K_1(A^d)^{2k} + \Gamma^k K_2\right), \ \textit{for } k = 0, 1, \quad (3.11)$$

$$K_1 = \sum_{i=0}^{\mu_1 - 1} D^\pi (D^2 + CB)^i C (A^d)^{2i+4} - \sum_{i=0}^{\mu_1} \Gamma (CB)^i C (A^d)^{2i+2}, \quad (3.12)$$

$$K_2 = \sum_{i=0}^{\nu_1 - 1} \Gamma^{i+2} C A^{2i} A^\pi, \quad (3.13)$$

$\nu_1 = \operatorname{ind}(A^2)$, $\mu_1 = \operatorname{ind}(D^2 + CB)$ *and* Γ *is defined by* (2.10).

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}.$$

According to assumptions of the theorem, we have that $PQP = 0$ and $PQ^2 = 0$. Hence we can apply Lemma 2.1 and we have

$$(P + Q)^d = Y_1 + Y_2 + \left(Y_1 (P^d)^2 + (Q^d)^2 Y_2 - Q^d (P^d)^2 - (Q^d)^2 P^d\right) PQ, \quad (3.14)$$

where Y_1 and Y_2 are defined by (2.1). Since $CBD = 0$, matrix Q satisfies condition of Lemma 2.7. After applying Lemma 2.7 and facts from Remark 4 we get

$$Y_1 = \begin{bmatrix} A^d + B(K_1 A + D K_1) & 0 \\ \Gamma C - \Gamma C A^\pi + (K_1 A + D K_1) A & 0 \end{bmatrix}, \quad (3.15)$$

$$Y_2 = \begin{bmatrix} B(K_2 A + D K_2) & B\Gamma \\ \Gamma C A^\pi + (K_2 A + D K_2) A & D\Gamma \end{bmatrix}, \quad (3.16)$$

where K_1 and K_2 are given in (3.12) and (3.13), respectively. Now, by substituting (3.16) and (3.15) into (3.14) we get that (3.10) holds. □

Notice that Theorem 3.4 is also an extension of a case when $ABC = 0$ and $BD = 0$ (see [19]).

The following result is direct corollary of Theorem 3.4.

Corollary 3.4 *Let* M *be given by* (1.1). *If* $CBD = 0$ *and* $AB = 0$ *then*

$$M^d = \begin{bmatrix} A^d + B\Theta_0 & B\Gamma \\ \Gamma C + \Theta_0 A & D\Gamma \end{bmatrix},$$

where Γ *and* Θ_0 *are defined by* (2.10) *and* (3.11) *respectively.*

As another extension of a result from [12], where formula for M^d is given under conditions $AB = 0$ and $CB = 0$, we offer the following theorem and its corollary.

Theorem 3.5 *Let M be matrix of a form* (1.1)*. If $ABC = 0$, $ABD = 0$ and $DCB = 0$ then*

$$M^d = \begin{bmatrix} A^d + B(N_1 + N_2) & \begin{array}{c} B\Psi + B(N_1(A^d)^2 + \Psi N_2)AB \\ +(A^d)^2B - B\Psi^2(CA + DC)A^dB \end{array} \\ \Psi C + (N_1 + N_2)A & \Psi D + (N_1 + N_2)B \end{bmatrix}, \tag{3.17}$$

where

$$N_1 = \sum_{i=0}^{\mu_1-1} (CB)^\pi (D^2+CB)^i (CA+DC)(A^d)^{2i+4} - \sum_{i=0}^{\mu_1} \Psi D^{2i}(CA+DC)(A^d)^{2i+2}, \tag{3.18}$$

$$N_2 = \sum_{i=0}^{\nu_1-1} \Psi^{i+2}(CA + DC)A^{2i}A^\pi, \tag{3.19}$$

$\nu_1 = \mathrm{ind}(A^2)$, $\mu_1 = \mathrm{ind}(D^2 + CB)$ *and Ψ is defined by* (2.6)*.*

Proof. Using the splitting

$$M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} := P + Q,$$

we get that matrices P and Q satisfy the conditions of Lemma 2.1. Furthermore, matrix Q satisfies the conditions of Lemma 2.6. After applying these lemmas, using Remark 3 and computing, we get that (3.17) holds. □

Next corollary follows immediately from Theorem 3.5.

Corollary 3.5 *Let M be given by* (1.1)*. If $DCB = 0$ and $AB = 0$ then*

$$M^d = \begin{bmatrix} A^d + B(N_1 + N_2) & B\Psi \\ \Psi C + (N_1 + N_2)A & \Psi D \end{bmatrix},$$

where Ψ, N_1 and N_2 are defined by (2.6)*,* (3.18) *and* (3.19)*, respectively.*

Cvetković and Milovanović (see [17]) offered a representation for M^d under conditions $ABC = 0$, $DC = 0$, with third condition $BD = 0$ (or BC is nilpotent, or D is nilpotent). Cvetković - Ilić (see [18]) extended this result and gave a formula for M^d under conditions $ABC = 0$ and $DC = 0$, without any additional condition. In our next result we replace second condition $DC = 0$ from [18] with two weaker conditions. Therefore, we can get results from [17, 18] as direct corollaries.

Theorem 3.6 *Let M be matrix of a form* (1.1)*, such that $ABC = 0$, $DCA = 0$ and $DCB = 0$. Then*

$$M^d = \begin{bmatrix} \Phi A + (U_1 + U_2)C & \Phi B + (U_1 + U_2)D \\ \begin{array}{c} C\Phi + C(U_1(D^d)^2 + \Phi U_2)DC \\ +(D^d)^2 C - C\Phi^2(AB + BD)D^d C \end{array} & D^d + C(U_1 + U_2) \end{bmatrix},$$

where

$$U_1 = \sum_{i=0}^{\mu_2 - 1} (BC)^\pi (A^2 + BC)^i (AB + BD)(D^d)^{2i+4} - \sum_{i=0}^{\mu_2} \Phi A^{2i}(AB + BD)(D^d)^{2i+2}$$

$$U_2 = \sum_{i=0}^{\nu_2 - 1} \Phi^{i+2}(AB + BD)D^{2i}D^\pi,$$

$\nu_2 = \mathrm{ind}(D^2)$, $\mu_2 = \mathrm{ind}(A^2 + BC)$ *and Φ is defined by* (2.3)*.*

Proof. If we split matrix M as

$$M = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} := P + Q,$$

we have $PQP = 0$ and $PQ^2 = 0$. Also, matrix P satisfies conditions of Lemma 2.4. After applying Lemma 2.1, Lemma 2.4, Remark 1 and computing we get that the statement of the theorem is valid. □

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