Some properties of (b, c)-inverses in rings

Jelena Višnjić[a,*], Ivana Stanišev[b], Yuanyuan Ke[c]

[a] Faculty of Medicine, University of Niš, 18000 Niš, Republic of Serbia;
[b] Technical Faculty in Bor, University of Belgrade, 19210 Bor, Republic of Serbia;
[c] School of Artificial Intelligence, Jianghan University, Wuhan, 430056, P. R. China

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Abstract

In this paper, we study (b, c)-invertibility of some arbitrary (b, c)-inverse and we give an equivalent condition for its (b, c)-invertibility. Moreover, we investigate when some well known properties of generalized inverses hold for (b, c)-inverses. Further, we study when the (b, c)-inverse of the unity is equal to the unity itself and we obtain equivalent conditions for this to hold. In addition, we investigate the reverse order law for the (b, c)-inverse. Finally, we prove that in the case when the unity is (b, c)-invertible, the set of all (b, c)-inverses, with respect to multiplication, is a group.

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1 Introduction

Throughout this paper we assume that R is a ring with the unity 1. In 2012, Drazin [5] introduced the concept of the (b, c)-inverse, as an outer inverse which generalizes some well known generalized inverses. Namely, for given $b, c \in R$, an element $a \in R$ is said to be (b, c)-invertible if there exists $y \in R$ such that

$$y \in (bRy) \cap (yRc), \quad yab = b, \ cay = c.$$

If such y exists, it is unique and it is called the (b, c)-inverse of a, denoted by $y = a^{(b,c)}$. Also, the (b, c)-inverse of a satisfies $yay = y$. By $R^{(b,c)}$ we will denote the set of all (b, c)-invertible elements of a ring R. After the publication [5] of Drazin, many authors have studied properties of (b, c)-inverses and its special cases and there are plenty of papers on this subject (for example, see [3], [4], [7]–[16]).

For the convenience of the reader, some fundamental concepts are given as follows. The set of all invertible elements of a ring R will be denoted by R^{-1}. Also, we denote by R_l^{-1} (R_r^{-1}) the set of all left (right) invertible elements of a ring R. Let us recall that an element $a \in R$ is regular if there exists $y \in R$ such that $aya = a$. Any inner inverse of a will be denoted by a^-. The set of all regular elements of a ring R will be denoted by R^-. The set of all inner inverses of $a \in R$ will be denoted by $a\{1\}$.

*Corresponding author: jelena.visnjic@medfak.ni.ac.rs

For $a \in R$, we define image ideals aR and Ra by

$$aR = \{ax : x \in R\}, \quad Ra = \{xa : x \in R\},$$

and kernel ideals a° and ${}^\circ a$ by

$$a^\circ = \{x \in R : ax = 0\}, \quad {}^\circ a = \{x \in R : xa = 0\}.$$

Let us recall that the Drazin inverse a^d [6] of an element $a \in R$ is the unique solution, if any, of equations:

$$a^k xa = a^k, \qquad xax = a, \qquad ax = xa,$$

for some $k \in \mathbb{N}$. The set of all Drazin invertible elements of a ring R will be denoted by R^d.

This paper is organized in the following way. In the second section of this paper we list some lemmas, which we will apply to derive our results. The third section of this paper contains results that we have obtained.

2 Auxiliary lemmas

Now, we will give some auxiliary lemmas, which will be useful in order to prove our results.

Lemma 2.1. ([5, Theorem 2.2]) *For $a, b, c \in R$, the following statements are equivalent:*

(i) *$a^{(b,c)}$ exists.*

(ii) *$b \in Rcab$ and $c \in cabR$.*

Lemma 2.2. ([3, Lemma 2.1]) *Let $a, b, c \in R$. The following statements are equivalent:*

(i) *$a^{(b,c)}$ exists;*

(ii) *$b, c \in R^-$ and there exists $y \in R$ such that $yab = b$, $cay = c$, $y = bb^-y = yc^-c$.*

In that case, $y = a^{(b,c)}$.

Lemma 2.3. ([8, Theorem 2.9]) *Let $a, b, c \in R$. The following statements are equivalent:*

(i) *$a \in R^{(b,c)}$;*

(ii) *there exists $y \in R$ such that $yay = y$, $yR = bR$ and $Ry = Rc$;*

(ii)′ *$b \in R^-$ and there exists $y \in R$ such that $yay = y$, ${}^\circ y = {}^\circ b$ and $Ry = Rc$;*

(ii)″ *$c \in R^-$ and there exists $y \in R$ such that $yay = y$, $yR = bR$ and $y^\circ = c^\circ$;*

(ii)‴ *$b, c \in R^-$ and there exists $y \in R$ such that $yay = y$, ${}^\circ y = {}^\circ b$ and $y^\circ = c^\circ$;*

(iii) *$b \in R^-$, ${}^\circ a \cap Rc = \{0\}$ and $R = Rca \oplus {}^\circ b$;*

(iii)′ *$c \in R^-$, $a^\circ \cap bR = \{0\}$ and $R = abR \oplus c^\circ$.*

In this case, $y = a^{(b,c)}$.

Lemma 2.4. ([16, Theorem 3.1]) *Let $a, b, c \in R$. The following statements are equivalent:*

(i) $a \in R^{(b,c)}$;

(ii) $c \in R^{-}$, $b^{\circ} = (ab)^{\circ}$, $R = abR \oplus c^{\circ}$;

(iii) $b \in R^{-}$, ${}^{\circ}c = {}^{\circ}(ca)$, $R = Rca \oplus {}^{\circ}b$.

In view of the item (ii) of this lemma, for $b = c$ we have that a is invertible along b and $a^{(b,b)} = a^{||b}$. See [9].

Lemma 2.5. ([14, Lemma 4.1]) *Let $b, c \in R$ and $a, d \in R^{(b,c)}$. Then $d^{(b,c)} = d^{(b,c)}aa^{(b,c)} = a^{(b,c)}ad^{(b,c)}$.*

Lemma 2.6. ([11, Theorem 3.3]) *Let $b, c, d \in R$ and $a \in R^{(b,c)}$. Then $cab = cdb$ if and only if $d \in R^{(b,c)}$ and $a^{(b,c)} = d^{(b,c)}$.*

Lemma 2.7. ([11, Theorem 4.4]) *Let $b, c \in R$ and $a \in R^{(b,c)}$. The following are equivalent:*

(i) $aa^{(b,c)} \in R^{(b,c)}$;

(ii) $a^{(b,c)}a \in R^{(b,c)}$;

(iii) $1 \in R^{(b,c)}$.

Moreover, if any of the above conditions is satisfied, then:

(a) $\left(aa^{(b,c)}\right)^{(b,c)} = \left(a^{(b,c)}a\right)^{(b,c)} = 1^{(b,c)}$;

(b) $1^{(b,c)} \in R^{(b,c)}$ *and* $\left(1^{(b,c)}\right)^{(b,c)} = 1^{(b,c)}$;

(c) $a^{(b,c)} = a^{(b,c)} \cdot 1^{(b,c)} = 1^{(b,c)} \cdot a^{(b,c)}$, *for every* $a \in R^{(b,c)}$.

Lemma 2.8. [14, Theorem 4.4] *Let $b, c \in R$ and $a, d \in R^{(b,c)}$. Then the following statements are equivalent:*

(i) $ad \in R^{(b,c)}$ *and* $(ad)^{(b,c)} = d^{(b,c)}a^{(b,c)}$;

(ii) $d^{(b,c)} = d^{(b,c)}add^{(b,c)}a^{(b,c)} = d^{(b,c)}a^{(b,c)}add^{(b,c)}$;

(iii) $a^{(b,c)} = a^{(b,c)}add^{(b,c)}a^{(b,c)} = d^{(b,c)}a^{(b,c)}ada^{(b,c)}$.

3 Results

It is well known that for the Drazin inverse of some element $a \in R^d$, we have that $(a^d)^d = aa^da$ and $((a^d)^d)^d = a^d$ hold [1]. It is natural to ask if these properties are also valid for (b, c)-inverses, so we investigate this question in what follows.

Theorem 3.1. *Let $b, c \in R$ and $a \in R^{(b,c)}$. The following statements are equivalent:*

(i) $a^{(b,c)} \in R^{(b,c)}$;

(ii) $1 \in R^{(b,c)}$.

If any of the above conditions is satisfied, then:

(a) $\left(a^{(b,c)}\right)^{(b,c)} = 1^{(b,c)}a1^{(b,c)}$;

(b) $\left(a^{(b,c)}\right)^{(b,c)} \in R^{(b,c)}$ *and* $\left(\left(a^{(b,c)}\right)^{(b,c)}\right)^{(b,c)} = a^{(b,c)}$

(c) $a^{(b,c)}\left(a^{(b,c)}\right)^{(b,c)} = \left(a^{(b,c)}\right)^{(b,c)}a^{(b,c)} = 1^{(b,c)}$.

Proof. Let the assumptions of the theorem hold. Denote by $y = a^{(b,c)}$.
(i) $\Rightarrow$ (ii): Let $y \in R^{(b,c)}$. We will prove that $R = Rc \oplus {}^\circ b$. Since $a, y \in R^{(b,c)}$, by Lemma 2.3 (ii)' and (iii), we have that ${}^\circ y = {}^\circ b$, $Ry = Rc$ and

$$\{0\} = {}^\circ y \cap Rc = {}^\circ b \cap Rc = {}^\circ y \cap Ry. \tag{1}$$

Further, since $y \in R^{(b,c)}$, by Lemma 2.3 (iii), we have:

$$R = Rcy \oplus {}^\circ b = Ry^2 \oplus {}^\circ y \subseteq Ry + {}^\circ y. \tag{2}$$

Now, from (1) and (2), it follows that $R = Ry \oplus^\circ y = Rc \oplus {}^\circ b$. By Lemma 2.4 (iii), it follows that $1 \in R^{(b,c)}$.
(ii) $\Rightarrow$ (i): Let $1 \in R^{(b,c)}$. We will prove that ${}^\circ y \cap Rc = \{0\}$ and $R = Rcy \oplus^\circ b$. Since $a \in R^{(b,c)}$, by Lemma 2.3, we have that ${}^\circ y = {}^\circ b$ and $Ry = Rc$. Further, from $1 \in R^{(b,c)}$ and Lemma 2.3 (iii) we have:

$$R = Rc \cdot 1 \oplus {}^\circ b = Rc \oplus {}^\circ y = Ry \oplus {}^\circ y. \tag{3}$$

Therefore,

$${}^\circ y \cap Rc = \{0\}. \tag{4}$$

Moreover, from $1 \in R^{(b,c)}$ and by Lemma 2.1, we have $b \in Rc \cdot 1 \cdot b$, i.e. there exists some $u \in R$, such that $b = ucb$. Let $r = 1 - uc$. Then $rb = b - ucb$, i.e. $r \in {}^\circ b = {}^\circ y$. Hence, $ry = 0$. Therefore, $y = 1 \cdot y = (r + uc)y = ucy \in Rcy$. Since $Rcy = Ry^2$, we have $y \in Ry^2$ and thereby $Ry \subseteq Ry^2$. Trivially, $Ry^2 \subseteq Ry$. Therefore, $Ry = Ry^2 = Rcy$. Now, from (3), we have that

$$R = Ry^2 \oplus {}^\circ y = Rcy \oplus {}^\circ b. \tag{5}$$

Now by (4), (5) and Lemma 2.3 (iii), we get that $y \in R^{(b,c)}$.

Now, let us prove the second part of the theorem. Let $a^{(b,c)} \in R^{(b,c)}$. By the previous proved part of this theorem $(i) \Leftrightarrow (ii)$, this means that $1 \in R^{(b,c)}$.
(a) Let $z = 1^{(b,c)}a1^{(b,c)}$. We will prove that z is the (b,c)-inverse of y. Using Lemma 2.7 (c), we get $1^{(b,c)}a^{(b,c)} = a^{(b,c)}$. Further, by Lemma 2.5, we have $1^{(b,c)}aa^{(b,c)} = 1^{(b,c)}$. Moreover, by definition of the (b,c)-inverse, we have $1^{(b,c)} \cdot 1 \cdot b = b$. Therefore, we get:

$$za^{(b,c)}b = 1^{(b,c)}a(1^{(b,c)}a^{(b,c)})b = (1^{(b,c)}aa^{(b,c)})b = 1^{(b,c)}b = b.$$

Similarly, as $a^{(b,c)}1^{(b,c)} = a^{(b,c)}$, $a^{(b,c)}a1^{(b,c)} = 1^{(b,c)}$ and $c1^{(b,c)} = c$, we have

$$ca^{(b,c)}z = c(a^{(b,c)}1^{(b,c)})a1^{(b,c)} = c(a^{(b,c)}a1^{(b,c)}) = c1^{(b,c)} = c.$$

Further, we have that $z \in bRz \cap zRc$. Indeed, since $1 \in R^{(b,c)}$, we have that $1^{(b,c)} \in bR1^{(b,c)} \cap 1^{(b,c)}Rc$. Therefore,

$$z = 1^{(b,c)}a1^{(b,c)} \in bR1^{(b,c)}a1^{(b,c)} = bRz,$$

and

$$z = 1^{(b,c)}a1^{(b,c)} \in 1^{(b,c)}a1^{(b,c)}Rc = zRc.$$

Thus, $(a^{(b,c)})^{(b,c)} = z$, i.e. $(a^{(b,c)})^{(b,c)} = 1^{(b,c)}a1^{(b,c)}$.

(b) Since $1 \in R^{(b,c)}$, we have $c \cdot 1 \cdot 1^{(b,c)} = c$ and $1^{(b,c)} \cdot 1 \cdot b = b$. Moreover, from the previous proved part (a) of this theorem, we have $(a^{(b,c)})^{(b,c)} = 1^{(b,c)}a1^{(b,c)}$. If we multiply the last equality by c from the left side and by b from the right side we get:

$$c(a^{(b,c)})^{(b,c)}b = c1^{(b,c)}a1^{(b,c)}b = cab.$$

By Lemma 2.6, we get that $(a^{(b,c)})^{(b,c)} \in R^{(b,c)}$ and $((a^{(b,c)})^{(b,c)})^{(b,c)} = a^{(b,c)}$.
(c) Using Lemma 2.5, we get:

$$1^{(b,c)} = 1^{(b,c)}a^{(b,c)}(a^{(b,c)})^{(b,c)} = (a^{(b,c)})^{(b,c)}a^{(b,c)}1^{(b,c)}.$$

By Lemma 2.7 (c), we have $1^{(b,c)}a^{(b,c)} = a^{(b,c)}1^{(b,c)} = a^{(b,c)}$. Hence, the statement (c) is valid. $\square$

Remark 3.1. *Let $b, c \in R$ and $1, a \in R^{(b,c)}$. Using Theorem 3.1 and by induction we get that:*

$$\overbrace{((a^{(b,c)})^{(b,c)}\dots)^{(b,c)}}^{k \text{ times}} \in R^{(b,c)}, \quad \textit{for every } k \in \mathbb{N}.$$

Furthermore:

$$\overbrace{((a^{(b,c)})^{(b,c)}\dots)^{(b,c)}}^{2k+1 \text{ times}} = a^{(b,c)}, \quad \textit{for } k \in \mathbb{N} \cup \{0\}$$

and

$$\overbrace{((a^{(b,c)})^{(b,c)}\dots)^{(b,c)}}^{2k \text{ times}} = (a^{(b,c)})^{(b,c)} = 1^{(b,c)}a1^{(b,c)}, \quad \textit{for } k \in \mathbb{N}.$$

As we have already mentioned, the Drazin inverse can be expressed as $(a^d)^d = aa^da$ [1]. In general, the analogous property does not hold for the (b,c)-inverse, as we can see in the following example.

Example 3.1. Let M_2 stands for the algebra of 2×2 complex matrices and let $a, b, c \in M_2$ be such that:

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix},$$

where $a_1 \neq 0$. By a straightforward computation, we get that the solution of the equations $yab = b$, $cay = c$ and $y = bb^-y = yc^-c$, is:

$$y = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, by Lemma 2.2, we have that a is (b,c)-invertible and that $a^{(b,c)} = y$. Moreover, we have that $a^{(b,c)}$ is also (b,c)-invertible and

$$(a^{(b,c)})^{(b,c)} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

However, we get that $aa^{(b,c)}a = \begin{bmatrix} a_1 & a_2 \\ a_3 & \frac{a_3a_2}{a_1} \end{bmatrix}$, which is not equal to $(a^{(b,c)})^{(b,c)}$ in general.

In the following theorem, we investigate equivalent conditions for $(a^{(b,c)})^{(b,c)} = aa^{(b,c)}a$ to be true. Also, we derive sufficient conditions for $(a^{(b,c)})^k = (a^k)^{(b,c)}$ to hold, for every $k \in \mathbb{N}$, which is also a well known property of the Drazin inverse.

Theorem 3.2. *Let $b, c \in R$ and $a \in R^{(b,c)}$. The following conditions are equivalent:*

(i) $aa^{(b,c)} = a^{(b,c)}a$;

(ii) $1 \in R^{(b,c)}$ *and* $a1^{(b,c)} = 1^{(b,c)}a$;

(iii) $1 \in R^{(b,c)}$ *and* $1^{(b,c)} = aa^{(b,c)} = a^{(b,c)}a$;

(iv) $a^{(b,c)} \in R^{(b,c)}$ *and* $\left(a^{(b,c)}\right)^{(b,c)} = aa^{(b,c)}a$.

Moreover, if any of the above conditions are satisfied, then $a^k \in R^{(b,c)}$ *and*

$$\left(a^k\right)^{(b,c)} = \left(a^{(b,c)}\right)^k, \quad \textit{for every} \;\; k \in \mathbb{N}.$$

Proof. Let the assumptions of the theorem hold.
(i) $\Rightarrow$ (iv): Let $aa^{(b,c)} = a^{(b,c)}a$ and denote by $y = aa^{(b,c)}a$. We will prove that y is the (b, c)-inverse of $a^{(b,c)}$. By Lemma 2.2, we have that $a^{(b,c)} = bb^-a^{(b,c)} = a^{(b,c)}c^-c$. We get:

$$ya^{(b,c)}b = aa^{(b,c)}aa^{(b,c)}b = a^{(b,c)}aa^{(b,c)}ab = a^{(b,c)}ab = b,$$

$$ca^{(b,c)}y = ca^{(b,c)}aa^{(b,c)}a = caa^{(b,c)}aa^{(b,c)} = caa^{(b,c)} = c,$$

$$y = a^{(b,c)}a^2 = bb^-a^{(b,c)}a^2 = bb^-y \quad \text{and} \quad y = a^2a^{(b,c)} = a^2a^{(b,c)}c^-c = yc^-c.$$

Hence, by Lemma 2.2, we get that $a^{(b,c)}$ is (b, c)-invertible and $(a^{(b,c)})^{(b,c)} = y$.
(iv) $\Rightarrow$ (iii): Since $a^{(b,c)} \in R^{(b,c)}$, by Theorem 3.1 we get that $1 \in R^{(b,c)}$. Also, by Theorem 3.1 (c), we have:

$$1^{(b,c)} = (a^{(b,c)})^{(b,c)}a^{(b,c)} = (aa^{(b,c)}a)a^{(b,c)} = aa^{(b,c)}$$

and

$$1^{(b,c)} = a^{(b,c)}(a^{(b,c)})^{(b,c)} = a^{(b,c)}(aa^{(b,c)}a) = a^{(b,c)}a.$$

(iii) $\Rightarrow$ (ii): Let $1 \in R^{(b,c)}$ and $1^{(b,c)} = aa^{(b,c)} = a^{(b,c)}a$. Then:

$$a1^{(b,c)} = aa^{(b,c)}a = (a^{(b,c)}a)a = 1^{(b,c)}a.$$

(ii) $\Rightarrow$ (i): Let $1 \in R^{(b,c)}$ and $a1^{(b,c)} = 1^{(b,c)}a$. By Lemma 2.7 (c), we have $a^{(b,c)} = 1^{(b,c)}a^{(b,c)} = a^{(b,c)}1^{(b,c)}$. Moreover, using Lemma 2.5, we get $1^{(b,c)} = 1^{(b,c)}aa^{(b,c)} = a^{(b,c)}a1^{(b,c)}$. Hence:

$$aa^{(b,c)} = a1^{(b,c)}a^{(b,c)} = 1^{(b,c)}aa^{(b,c)} = 1^{(b,c)},$$

$$a^{(b,c)}a = a^{(b,c)}1^{(b,c)}a = a^{(b,c)}a1^{(b,c)} = 1^{(b,c)}.$$

Therefore, $aa^{(b,c)} = a^{(b,c)}a$.

Now, let us prove the second part of the theorem. Namely, we will prove that $(a^{(b,c)})^2$ is the (b, c)-inverse of a^2. We have:

$$ca^2(a^{(b,c)})^2 = caa^{(b,c)}aa^{(b,c)} = caa^{(b,c)} = c,$$
$$(a^{(b,c)})^2a^2b = a^{(b,c)}aa^{(b,c)}ab = a^{(b,c)}ab = b,$$
$$bb^-(a^{(b,c)})^2 = (bb^-a^{(b,c)})a^{(b,c)} = (a^{(b,c)})^2$$

and

$$(a^{(b,c)})^2c^-c = a^{(b,c)}(a^{(b,c)}c^-c) = (a^{(b,c)})^2.$$

By Lemma 2.2, we get that $(a^2)^{(b,c)} = (a^{(b,c)})^2$. Trivially, by induction, we get $a^k \in R^{(b,c)}$ and $(a^k)^{(b,c)} = (a^{(b,c)})^k$, for every $k \in \mathbb{N}$. $\square$

Remark 3.2. We give the following three remarks concerning the previously proved Theorem 3.2.

(i) In [4], Drazin has studied the EP property of (b, c)-invertible matrices. Recently, Wang, Mosić and Yao [12], considered the EP property of the (b, c)-inverse in a semigroup with a unity. Namely, the (b, c)-inverse $a^{(b,c)}$ of a satisfies the EP property, if $aa^{(b,c)} = a^{(b,c)}a$. We remark that in Theorem 3.2, we derived equivalent conditions for the EP property of the (b, c)-inverse to hold in a ring with a unity. For specific EP properties for the inverse along an element, see Section 7 of [2].

(ii) If $(a^k)^{(b,c)} = (a^{(b,c)})^k$, for every $k \in \mathbb{N}$, then $aa^{(b,c)} = a^{(b,c)}a$ does not have to be true in general. Indeed, let $b, c \in M_2$ be defined as in Example 3.1 and let $a \in M_2$ be such that:

$$a = \begin{bmatrix} a_1 & 0 \\ a_3 & a_4 \end{bmatrix},$$

where $a_1 \neq 0$ and $a_3 \neq 0$. By Example 3.1, we have that a and a^2 are both (b, c)-invertible and:

$$a^{(b,c)} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad (a^2)^{(b,c)} = \begin{bmatrix} (a_1^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix} = (a^{(b,c)})^2.$$

By induction, we get that a^k is (b, c)-invertible and $(a^k)^{(b,c)} = (a^{(b,c)})^k$, for every $k \in \mathbb{N}$. Moreover,

$$aa^{(b,c)} = \begin{bmatrix} 1 & 0 \\ a_3a_1^{-1} & 0 \end{bmatrix} \quad \text{and} \quad a^{(b,c)}a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

Hence, $aa^{(b,c)} \neq a^{(b,c)}a$.

(iii) In general case, if $a^{(b,c)}a = 1^{(b,c)}$, then $aa^{(b,c)} = 1^{(b,c)}$ does not have to be true (also, in general case, from $aa^{(b,c)} = 1^{(b,c)}$ it does not follow that $a^{(b,c)}a = 1^{(b,c)}$). Indeed, consider matrices $b, c, a \in M_2$, defined as in the part (ii) of this remark. We have that the unity $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is (b, c)-invertible and $u^{(b,c)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Now, from (6), we have that $a^{(b,c)}a = u^{(b,c)}$, but $aa^{(b,c)} \neq u^{(b,c)}$.

In [11, Theorem 3.7], authors proved that if $a \in R^{(b,c)}$, then $aa^{(b,c)}a \in R^{(b,c)}$ and $(aa^{(b,c)}a)^{(b,c)} = a^{(b,c)}$. In the next theorem, we study some generalizations of the mentioned case.

Theorem 3.3. *Let $b, c \in R$ and $a, d \in R^{(b,c)}$. Then $aa^{(b,c)}d$ and $da^{(b,c)}a$ are (b, c)-invertible and:*

$$\left(aa^{(b,c)}d\right)^{(b,c)} = \left(da^{(b,c)}a\right)^{(b,c)} = d^{(b,c)}.$$

Proof. Let the assumptions of the theorem hold. As $a, d \in R^{(b,c)}$, by Lemma 2.5, we have $d^{(b,c)}aa^{(b,c)} = a^{(b,c)}ad^{(b,c)} = d^{(b,c)}$ and $a^{(b,c)}dd^{(b,c)} = d^{(b,c)}da^{(b,c)} = a^{(b,c)}$. We will prove that $d^{(b,c)}$ is the (b, c)-inverse of $aa^{(b,c)}d$.

$$d^{(b,c)}(aa^{(b,c)}d)b = d^{(b,c)}db = b \quad \text{and} \quad c(aa^{(b,c)}d)d^{(b,c)} = caa^{(b,c)} = c.$$

Moreover, since $d \in R^{(b,c)}$, we have $d^{(b,c)} \in bRd^{(b,c)} \cap d^{(b,c)}Rc$. Thus, $aa^{(b,c)}d$ is (b, c)-invertible and $(aa^{(b,c)}d)^{(b,c)} = d^{(b,c)}$.

Similarly, using Lemma 2.5, we prove that $d^{(b,c)}$ is the (b, c)-inverse of $da^{(b,c)}a$:

$$d^{(b,c)}(da^{(b,c)}a)b = a^{(b,c)}ab = b \quad \text{and} \quad c(da^{(b,c)}a)d^{(b,c)} = cdd^{(b,c)} = c.$$

□

In the following theorem, we study (b,c)-invertibility of an inner inverse of (b,c)-invertible element.

Theorem 3.4. *Let $b, c \in R$ and $a \in R^-$. Then the following statements hold:*

(i) *If $a \in R^{(b,c)}$, then $a^- \in R^{(ab,ca)}$ and $(a^-)^{(ab,ca)} = aa^{(b,c)}a$, for arbitrary $a^- \in a\{1\}$;*

(ii) *If $a^- \in R^{(ab,ca)}$, $bR \subseteq a^-aR$ and $Rc \subseteq Raa^-$, for arbitrary $a^- \in a\{1\}$, then $a \in R^{(b,c)}$ and $a^{(b,c)} = a^-(a^-)^{(ab,ca)}a^-$.*

Proof. Let the assumptions of the theorem hold.
(i) Let $a \in R^{(b,c)} \cap R^-$. We prove that $z = aa^{(b,c)}a$ is the (ab, ca)-inverse of arbitrary $a^- \in a\{1\}$. As $a \in R^{(b,c)}$, then $b = a^{(b,c)}ab$ and $c = caa^{(b,c)}$. Therefore,

$$ab = aa^{(b,c)}ab = aa^{(b,c)}aa^-ab = za^-ab,$$

as well as

$$ca = caa^{(b,c)}a = caa^-aa^{(b,c)}a = caa^-z.$$

Since $a \in R^{(b,c)}$, then $a^{(b,c)} = bta^{(b,c)} = a^{(b,c)}pc$, for some $t, p \in R$. Thus,

$$z = aa^{(b,c)}a = abta^{(b,c)}a = abta^{(b,c)}aa^{(b,c)}a = abta^{(b,c)}z \in abRz.$$

Also, we have

$$z = aa^{(b,c)}a = aa^{(b,c)}pca = aa^{(b,c)}aa^{(b,c)}pca \in zRca.$$

Therefore,

$$z \in (abRz) \cap (zRca).$$

So we get that $a^- \in R^{(b,c)}$ and $(a^-)^{(ab,ca)} = aa^{(b,c)}a$.
(ii) Let $bR \subseteq a^-aR$ and $Rc \subseteq Raa^-$, for arbitrary $a^- \in a\{1\}$. There exist some $\mu, \omega \in R$, such that $b = a^-a\mu$ and $c = \omega aa^-$. Thus, we have $b = a^-a\mu = a^-aa^-a\mu = a^-ab$, as well as $c = \omega aa^- = \omega aa^-aa^- = caa^-$. Let $a^- \in R^{(ab,ca)}$ and denote by $x = (a^-)^{(ab,ca)}$. We show that $y = a^-xa^-$ is the (b,c)-inverse of a. As $a^- \in R^{(ab,ca)}$, we have $ab = xa^-ab$ and $ca = caa^-x$. Hence,

$$yab = a^-(xa^-ab) = a^-ab = b \quad \text{and} \quad cay = (caa^-x)a^- = caa^- = c.$$

Also, $x \in abRx \cap xRca$, so there exist some $u, v \in R$ such that $x = abux = xvca$. Therefore,

$$y = a^-xa^- = (a^-ab)uxa^- = buxa^- = bux(a^-xa^-) = buxy \in bRy$$

and

$$y = a^-xa^- = a^-xv(caa^-) = a^-xvc = a^-(xa^-x)vc = yxvc \in yRc,$$

so $y \in bRy \cap yRc$. Thus, $a \in R^{(b,c)}$ and $a^{(b,c)} = y = a^-xa^-$. $\square$

Using previously proved theorems, we derive some properties of (b,c)-invertible elements in the next theorem.

Theorem 3.5. *Let $b, c \in R$ and $a \in R^{(b,c)}$. The following statements hold.*

(i) $a = aa^{(b,c)}a + u$, *for some* $u \in {}^\circ b \cap c^\circ$.

(i)' *If* $1 \in R^{(b,c)}$, *then* $1 = 1^{(b,c)} + u$, *for some* $u \in {}^\circ b \cap c^\circ$.

(ii) *If* $1 \in R^{(b,c)}$, *then* $(a^{(b,c)})^{(b,c)} = ((a^{(b,c)})^2a)^{(b,c)} = (a(a^{(b,c)})^2)^{(b,c)}$.

(iii) *If* $1 \in R^{(b,c)}$, *then* $1^{(b,c)} = (a(a^{(b,c)})^2a)^{(b,c)}$.

(iv) *If* $1 \in R^{(b,c)}$, *then* $(a1^{(b,c)})^{(b,c)} = (1^{(b,c)}a)^{(b,c)} = a^{(b,c)}$.

(v) *If* $^\circ b \cap c^\circ = \{0\}$, *then* $a^{(b,c)} \in a\{1\}$, *i.e.* $a = aa^{(b,c)}a$.

(vi) *If* $^\circ b \cap c^\circ = \{0\}$, *then* $(a^{(b,c)})^{(ab,ca)} = a$.

(vii) $1 = a^{(b,c)}a + u$, *for some* $u \in {}^\circ b$, *and* $1 = aa^{(b,c)} + v$, *for some* $v \in c^\circ$.

Proof. Let the assumptions of the theorem hold.

(i) Since $a \in R^{(b,c)}$, by Lemma 2.3 (iii), it follows that $R = Rca \oplus {}^\circ b$. Therefore, $a \in R$ can be written in the form

$$a = tca + u, \tag{7}$$

for some $t \in R$ and some $u \in {}^\circ b$. The multiplication of (7) by $a^{(b,c)}$ from the right side gives:

$$aa^{(b,c)} = tcaa^{(b,c)} + ua^{(b,c)}.$$

By Lemma 2.3, we have that ${}^\circ b = {}^\circ(a^{(b,c)})$ and thereby $ua^{(b,c)} = 0$. Moreover, $caa^{(b,c)} = c$ and therefore we get that $tc = aa^{(b,c)}$. If we return it back into the equation (7) we get

$$a = aa^{(b,c)}a + u, \quad u \in {}^\circ b. \tag{8}$$

If we multiply (8) by c from the left side, we achieve $cu = 0$, i.e. $u \in c^\circ$. Thus, $u \in {}^\circ b \cap c^\circ$.

(i)' Follows by the previous proved part (i), for the case of $a = 1$.

(ii) Since $1 \in R^{(b,c)}$, by Theorem 3.1 we have that $a^{(b,c)} \in R^{(b,c)}$. Further, by Theorem 3.3, we have that for $a, d \in R^{(b,c)}$, $aa^{(b,c)}d, da^{(b,c)}a \in R^{(b,c)}$ and $(aa^{(b,c)}d)^{(b,c)} = (da^{(b,c)}a)^{(b,c)} = d^{(b,c)}$. Thus, taking $d = a^{(b,c)}$, we get that statement (ii) is true.

(iii) From $1 \in R^{(b,c)}$ and Lemma 2.7, it follows that $aa^{(b,c)} \in R^{(b,c)}$ and $(aa^{(b,c)})^{(b,c)} = 1^{(b,c)}$. Now, after applying Theorem 3.3, i.e. $(da^{(b,c)}a)^{(b,c)} = d^{(b,c)}$ for $d = aa^{(b,c)}$, we get that the statement (iii) is valid.

(iv) By Theorem 3.3, we have that for $a, d \in R^{(b,c)}$, $\left(dd^{(b,c)}a\right)^{(b,c)} = \left(ad^{(b,c)}d\right)^{(b,c)} = a^{(b,c)}$. Now, taking $d = 1$, we get that (iv) holds.

(v) From the part (i) of this theorem, we have that $a = aa^{(b,c)}a + u$, $\quad u \in {}^\circ b \cap c^\circ$. If ${}^\circ b \cap c^\circ = \{0\}$, then $u \in {}^\circ b \cap c^\circ = \{0\}$, so $u = 0$. Thus, $a = aa^{(b,c)}a$, i.e. $a^{(b,c)} \in a\{1\}$.

(vi) From the part (v) of this theorem, we have that $a = aa^{(b,c)}a$, i.e. $a^{(b,c)} \in a\{1\}$. Now, using Theorem 3.4 (i.e. $(a^-)^{(ab,ca)} = aa^{(b,c)}a$, for arbitrary $a^- \in a\{1\}$), we have that $(a^{(b,c)})^{(ab,ca)} = aa^{(b,c)}a = a$.

(vii) Let $a \in R^{(b,c)}$. By Lemma 2.3, it follows that $R = Rca \oplus {}^\circ b$. As $1 \in R$, it follows that

$$1 = pca + u, \text{ for some } p \in R \text{ and } u \in {}^\circ b. \tag{9}$$

If we multiply (9) by $a^{(b,c)}$ from the right side, and using the fact that $u \in {}^\circ b = {}^\circ(a^{(b,c)})$ and $caa^{(b,c)} = c$, we have that

$$a^{(b,c)} = pcaa^{(b,c)} + ua^{(b,c)} = pc.$$

If we return it back in the equation (9), we get that $1 = a^{(b,c)}a + u$, for some $u \in {}^\circ b$. In a similar way, using Lemma 2.3 and $R = abR \oplus c^\circ$, we can prove that $1 = aa^{(b,c)} + v$, for some $v \in c^\circ$, also holds. $\square$

In [11], authors showed that in general case $1^{(b,c)} = 1$ does not hold. As it was stated by Drazin [5], if we take $b = c = 1$, the (b,c)-inverse reduces to classical inverse. Hence, $1^{(b,c)} = 1$ for $b = c = 1$. However, b (and also c) does not have to be equal to 1, for $1^{(b,c)} = 1$ to hold.

Example 3.2. Let $b, c \in M_2$ be such that:

$$b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the unity $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is (b, c)-invertible and $u^{(b,c)} = u$ is satisfied, but $b \neq u$ and $c \neq u$.

In the following theorem we prove that $1 \in R^{(b,c)}$ and $1^{(b,c)} = 1$ is actually equivalent to $R^{(b,c)} = R^{-1}$ and $a^{(b,c)} = a^{-1}$, for arbitrary $a \in R^{(b,c)}$. Also, we give another equivalent condition, concerning b and c, for $1^{(b,c)} = 1$ to be valid.

Theorem 3.6. *Let $a, b, c \in R$. The following conditions are equivalent:*

(i) $1 \in R^{(b,c)}$ *and* $1^{(b,c)} = 1$*;*

(ii) $b \in R_r^{-1}$ *and* $c \in R_l^{-1}$*;*

(iii) $R^{(b,c)} = R^{-1}$ *and* $a^{(b,c)} = a^{-1}$*, for every* $a \in R^{(b,c)}$*.*

Proof. (i) $\Rightarrow$ (ii): Let $1 \in R^{(b,c)}$ and $1^{(b,c)} = 1$. By Lemma 2.2, $1^{(b,c)} = bb^{-}1^{(b,c)} = 1^{(b,c)}c^{-}c$ and therefore $1 = bb^{-} = c^{-}c$. Hence, b is right invertible and c is left invertible.
(ii) $\Rightarrow$ (i): Let $b \in R_r^{-1}$ and $c \in R_l^{-1}$. We will prove that Lemma 2.3 (ii)$'''$ holds for $a = 1$ and $y = 1$. Trivially, $yay = y$ holds. Further, since $bb_r^{-1} = 1$ and $c_l^{-1}c = 1$, we have that $bb_r^{-1}b = b$ and $cc_l^{-1}c = c$, i.e. $b, c \in R^{-}$. Now, let us prove that ${}^{\circ}b = \{0\} = {}^{\circ}1$ and $c^{\circ} = \{0\} = 1^{\circ}$. If $x \in^{\circ} b$, then $xb = 0$. Therefore, $xbb_r^{-1} = 0$, i.e. $x = 0$. Similarly, if $x \in c^{\circ}$, we have $cx = 0$ and thereby $c_l^{-1}cx = 0$, i.e. $x = 0$.
(i) $\Rightarrow$ (iii): Let $1 \in R^{(b,c)}$ and $1^{(b,c)} = 1$.

If $a \in R^{(b,c)}$, using Lemma 2.5 we get that $1^{(b,c)} = 1^{(b,c)}aa^{(b,c)} = a^{(b,c)}a1^{(b,c)}$, i.e. $1 = aa^{(b,c)} = a^{(b,c)}a$. Hence, $a \in R^{-1}$ and $a^{-1} = a^{(b,c)}$. Thus, $R^{(b,c)} \subseteq R^{-1}$.

If $a \in R^{-1}$, then $a^{-1}ab = b$ and $caa^{-1} = c$. Moreover, since $1^{(b,c)} = 1$, by Lemma 2.2 we have that $1 = bb^{-} = c^{-}c$. Hence, $a^{-1} = bb^{-}a^{-1} = a^{-1}c^{-}c$. Once more, using Lemma 2.2, we get that $a \in R^{(b,c)}$ and $a^{(b,c)} = a^{-1}$. Therefore, $R^{-1} \subseteq R^{(b,c)}$.
(iii) $\Rightarrow$ (i): Let $R^{-1} = R^{(b,c)}$ and $a^{(b,c)} = a^{-1}$, for every $a \in R^{(b,c)}$. Since $1 \in R^{-1}$, we get that (i) holds trivially. $\square$

In the following theorem we assume that $1 \in R^{(b,c)}$ and we give necessary and sufficient conditions for $1^{(b,c)} = 1$ to be true.

Theorem 3.7. *Let $b, c \in R$ and $1 \in R^{(b,c)}$. The following conditions are equivalent:*

(i) $1^{(b,c)} = 1$*;*

(ii) $1 \in bR \cup Rc$*;*

(iii) ${}^{\circ}b \cap c^{\circ} = \{0\}$*.*

Proof. Let the assumptions of the theorem hold. Since $1 \in R^{(b,c)}$, we have $1^{(b,c)} \in bR1^{(b,c)} \subseteq bR$ and $1^{(b,c)} \in 1^{(b,c)}Rc \subseteq Rc$, i.e. $1^{(b,c)} \in bR \cap Rc$.
(i) $\Rightarrow$ (ii): Let $1^{(b,c)} = 1$. Then $1 \in bR \cap Rc \subseteq bR \cup Rc$.
(ii) $\Rightarrow$ (i): Now assume that $1 \in bR \cup Rc$.

Let $1 \in Rc$. As $1 \in R^{(b,c)}$, then $b = 1^{(b,c)} \cdot 1 \cdot b = 1^{(b,c)} \cdot b$ and also $b = 1 \cdot b$. Thus, $1^{(b,c)} \cdot b - 1 \cdot b = 0$, i.e. $1^{(b,c)} - 1 \in {}^{\circ}b$. Since $1 \in Rc$, then $1^{(b,c)} - 1 \in Rc$. Hence, $1^{(b,c)} - 1 \in Rc \cap^{\circ} b$. By Lemma 2.3 (iii), we have that $R = Rc \oplus {}^{\circ}b$. Therefore, $1^{(b,c)} - 1 = 0$.

Let $1 \in bR$. Since $1 \in R^{(b,c)}$, we have $c = c \cdot 1 \cdot 1^{(b,c)} = c \cdot 1^{(b,c)}$ and $c = c \cdot 1$ and so $1^{(b,c)} - 1 \in c^{\circ}$. As $1 \in bR$, we get $1^{(b,c)} - 1 \in bR$. Therefore, $1^{(b,c)} - 1 \in bR \cap c^{\circ}$. By Lemma 2.3 (iii)$'$, we have $R = bR \oplus\ c^{\circ}$. Thus, it follows that $1^{(b,c)} - 1 = 0$.

(i) $\Rightarrow$ (iii): Let $1^{(b,c)} = 1$ and $x \in\ {}^{\circ}b$. Then we have $x = x \cdot 1 = x \cdot 1^{(b,c)} = 0$, as ${}^{\circ}b =\ {}^{\circ}(1^{(b,c)})$. Hence, ${}^{\circ}b = \{0\}$. Similarly, let $x \in c^{\circ}$. Since $c^{\circ} = (1^{(b,c)})^{\circ}$, we have $x = 1 \cdot x = 1^{(b,c)} \cdot x = 0$. Therefore, $c^{\circ} = \{0\}$. Thereby, ${}^{\circ}b \cap c^{\circ} = \{0\}$.

(iii) $\Rightarrow$ (i): Let ${}^{\circ}b \cap c^{\circ} = \{0\}$. Applying Theorem 3.5 (v), for $a = 1$, we get that $1 = 1 \cdot 1^{(b,c)} \cdot 1 = 1^{(b,c)}$. □

In what follows, we investigate the reverse order law for the (b,c)-inverse in a ring with a unity. Namely, for $a, d \in R^{(b,c)}$, we study when $ad \in R^{(b,c)}$ and

$$(ad)^{(b,c)} = d^{(b,c)}a^{(b,c)}. \tag{10}$$

In [3] authors proved that (10) holds when $db = bd$ and $ac = ca$ and also when $ab = ba$ and $ac = ca$. Also, in [10] it was shown that (10) is valid when $ab = ba$ and $ac = ca$. Moreover, in [15], authors proved that (10) holds if $aa^{(b,c)} = a^{(b,c)}a$ and $dd^{(b,c)} = d^{(b,c)}d$. These results were generalized by Wang in [13], where the author proved that the equality (10) holds when the first factor of the product ad commutes with its (b,c)-inverse (i.e. when $aa^{(b,c)} = a^{(b,c)}a$ holds). In the next theorem we prove that (10) is also valid when the second factor of the product ad commutes with its (b,c)-inverse.

Theorem 3.8. *Let $b, c \in R$ and $a, d \in R^{(b,c)}$. Consider the conditions:*

(i) *$dd^{(b,c)} = d^{(b,c)}d$;*

(ii) *$aa^{(b,c)} = a^{(b,c)}a$.*

If any of the above conditions is satisfied, then $ad \in R^{(b,c)}$ and

$$(ad)^{(b,c)} = d^{(b,c)}a^{(b,c)}.$$

Proof. Let the assumptions of the theorem hold.
(i) Let $dd^{(b,c)} = d^{(b,c)}d$. By Theorem 3.2, it follows that $1 \in R^{(b,c)}$ and $1^{(b,c)} = dd^{(b,c)} = d^{(b,c)}d$. Also, by Lemma 2.7, $a^{(b,c)} = a^{(b,c)} \cdot 1^{(b,c)} = 1^{(b,c)} \cdot a^{(b,c)}$ holds for every $a \in R^{(b,c)}$. Moreover, by Lemma 2.5, $d^{(b,c)}aa^{(b,c)} = a^{(b,c)}ad^{(b,c)} = d^{(b,c)}$, for every $a, d \in R^{(b,c)}$. Now, we will apply Lemma 2.8. Namely, we will prove that $d^{(b,c)} = d^{(b,c)}add^{(b,c)}a^{(b,c)} = d^{(b,c)}a^{(b,c)}add^{(b,c)}$. We have:

$$d^{(b,c)}add^{(b,c)}a^{(b,c)} = d^{(b,c)}a1^{(b,c)}a^{(b,c)} = d^{(b,c)}aa^{(b,c)} = d^{(b,c)},$$

$$d^{(b,c)}a^{(b,c)}add^{(b,c)} = d^{(b,c)}a^{(b,c)}a1^{(b,c)} = d^{(b,c)}1^{(b,c)} = d^{(b,c)}.$$

Hence, by Lemma 2.8, this means that $ad \in R^{(b,c)}$ and $(ad)^{(b,c)} = d^{(b,c)}a^{(b,c)}$.
(ii) See [13, Corrolary 3.4]. Also, as in the proof of the part (i), using Lemma 2.8 (iii), it can be proved that (ii) holds.

□

For two characterizations of the reverse order law for the inverse along an element, see Section 6 of [2]. Now, we give the following result.

Theorem 3.9. *Let $b, c \in R$ and $a, d, 1 \in R^{(b,c)}$. Then $a1^{(b,c)}d \in R^{(b,c)}$ and:*

$$(a1^{(b,c)}d)^{(b,c)} = d^{(b,c)}a^{(b,c)}.$$

Proof. Denote by $y = d^{(b,c)}a^{(b,c)}$. Let us show that y is the (b,c)-inverse of $a1^{(b,c)}d$. Using Lemma 2.5, we get $1^{(b,c)}dd^{(b,c)} = 1^{(b,c)}$. Moreover, by Lemma 2.7 (c), we have $1^{(b,c)}a^{(b,c)} = a^{(b,c)}$. Thus:

$$ca1^{(b,c)}dy = ca(1^{(b,c)}dd^{(b,c)})a^{(b,c)} = ca(1^{(b,c)}a^{(b,c)}) = caa^{(b,c)} = c.$$

Further, by Lemma 2.5, we get $a^{(b,c)}a1^{(b,c)} = 1^{(b,c)}$. Also,by Lemma 2.7 (c), we have that $d^{(b,c)}1^{(b,c)} = d^{(b,c)}$. Hence:

$$ya1^{(b,c)}db = d^{(b,c)}(a^{(b,c)}a1^{(b,c)})db = (d^{(b,c)}1^{(b,c)})db = d^{(b,c)}db = b.$$

Moreover, using Lemma 2.2 (ii), we have $bb^-d^{(b,c)} = d^{(b,c)}$ and $a^{(b,c)}c^-c = a^{(b,c)}$. Therefore:

$$bb^-y = (bb^-d^{(b,c)})a^{(b,c)} = d^{(b,c)}a^{(b,c)} = y$$

and

$$yc^-c = d^{(b,c)}(a^{(b,c)}c^-c) = d^{(b,c)}a^{(b,c)} = y.$$

Applying Lemma 2.2, we get that $a1^{(b,c)}d$ is (b,c)-invertible and $(a1^{(b,c)}d)^{(b,c)} = d^{(b,c)}a^{(b,c)}$. □

Remark 3.3. From Theorem 3.9, it follows that for $a, d, 1 \in R^{(b,c)}$, if $a1^{(b,c)}d = ad$, then the reverse order law (10) holds.

As a corollary of Theorem 3.9, we get that, in the case when $1 \in R^{(b,c)}$, the reverse order law holds for the product of some arbitrary (b,c)-invertible element and some arbitrary (b,c)-inverse.

Corollary 3.1. *Let $b, c \in R$ and $a, d, 1 \in R^{(b,c)}$. Then $a^{(b,c)}d$, $da^{(b,c)}$, $d^{(b,c)}a$ and $ad^{(b,c)}$ are all (b,c)-invertible and the following hold:*

(i) $\left(a^{(b,c)}d\right)^{(b,c)} = d^{(b,c)}\left(a^{(b,c)}\right)^{(b,c)}$;

(ii) $\left(da^{(b,c)}\right)^{(b,c)} = \left(a^{(b,c)}\right)^{(b,c)} d^{(b,c)}$;

(iii) $\left(\left(a^{(b,c)}d\right)^{(b,c)}\right)^{(b,c)} = \left(d^{(b,c)}a\right)^{(b,c)} = a^{(b,c)}\left(d^{(b,c)}\right)^{(b,c)}$;

(iv) $\left(\left(da^{(b,c)}\right)^{(b,c)}\right)^{(b,c)} = \left(ad^{(b,c)}\right)^{(b,c)} = \left(d^{(b,c)}\right)^{(b,c)} a^{(b,c)}$.

Proof. Let the assumptions of the theorem hold.
(i) Since $1 \in R^{(b,c)}$, by Theorem 3.1 we have that $a^{(b,c)} \in R^{(b,c)}$. Also, by Lemma 2.7 (c), $a^{(b,c)}1^{(b,c)} = a^{(b,c)}$. Now, by Theorem 3.9, we have that $a^{(b,c)}1^{(b,c)}d$ is (b,c)-invertible and

$$(a^{(b,c)}d)^{(b,c)} = (a^{(b,c)}1^{(b,c)}d)^{(b,c)} = d^{(b,c)}(a^{(b,c)})^{(b,c)}.$$

(ii) In the same manner as in (i), we get that $da^{(b,c)} \in R^{(b,c)}$ and that (ii) is valid.
(iii) As it was proved in (i), $a^{(b,c)}d \in R^{(b,c)}$. Since $1 \in R^{(b,c)}$, by Theorem 3.1 it follows that $(a^{(b,c)}d)^{(b,c)}$ is also (b,c)-invertible. Further, by Theorem 3.1 (a) and Lemma 2.7 (c), we have:

$$\begin{aligned}((a^{(b,c)}d)^{(b,c)})^{(b,c)} &= 1^{(b,c)}a^{(b,c)}d1^{(b,c)} = a^{(b,c)}1^{(b,c)}d1^{(b,c)} = \\ &= a^{(b,c)}(d^{(b,c)})^{(b,c)}.\end{aligned}$$

Now, by the previous proved part (i) of this theorem, it follows that $a^{(b,c)}(d^{(b,c)})^{(b,c)} = (d^{(b,c)}a)^{(b,c)}$.
(iv) Using the similar method as in the proof of the part (iii) of this theorem, we get that the statement (iv) is valid. □

In order to prove our next result, we investigate equivalent conditions for the product of two (b,c)-inverses to be equal to $1^{(b,c)}$.

Theorem 3.10. *Let $b, c \in R$ and $1, a, d \in R^{(b,c)}$. The following conditions are equivalent:*

(i) $d^{(b,c)} = (a^{(b,c)})^{(b,c)}$,

(ii) $a^{(b,c)} = (d^{(b,c)})^{(b,c)}$,

(iii) $d^{(b,c)}a^{(b,c)} = 1^{(b,c)}$,

(iv) $a^{(b,c)}d^{(b,c)} = 1^{(b,c)}$.

Proof. Let the assumptions of the theorem hold.
(i) $\Rightarrow$ (iii): Let $d^{(b,c)} = (a^{(b,c)})^{(b,c)}$. By Theorem 3.1 (c), it follows that

$$d^{(b,c)}a^{(b,c)} = (a^{(b,c)})^{(b,c)}a^{(b,c)} = 1^{(b,c)}.$$

(iii) $\Rightarrow$ (ii): Let $d^{(b,c)}a^{(b,c)} = 1^{(b,c)}$. By Lemma 2.7 (c), we have $1^{(b,c)}a^{(b,c)} = a^{(b,c)}$. Further, by Lemma 2.5, $1^{(b,c)} = 1^{(b,c)}dd^{(b,c)}$ holds. Moreover, by Theorem 3.1 (a), we have $1^{(b,c)}d1^{(b,c)} = (d^{(b,c)})^{(b,c)}$. Using all of these facts, we get:

$$a^{(b,c)} = 1^{(b,c)}a^{(b,c)} = 1^{(b,c)}d(d^{(b,c)}a^{(b,c)}) = 1^{(b,c)}d1^{(b,c)} = (d^{(b,c)})^{(b,c)}.$$

(ii) $\Rightarrow$ (iv): Similarly as (i) $\Rightarrow$ (iii).
(iv) $\Rightarrow$ (i): Similarly as (iii) $\Rightarrow$ (ii). $\square$

In the following theorem we prove that, in the case when $1 \in R^{(b,c)}$, the set of all (b,c)-inverses of a ring R, with respect to operation "$\cdot$", is a group. Denote by $(R^{(b,c)})^{\otimes}$ the set of all (b,c)-inverses of elements of a ring R, i.e.:

$$(R^{(b,c)})^{\otimes} = \left\{ y \in R \mid y = a^{(b,c)}, \;\; a \in R^{(b,c)} \right\}.$$

Theorem 3.11. *Let $b, c \in R^{-}$ and let $1 \in R^{(b,c)}$. Then $\big((R^{(b,c)})^{\otimes}, \cdot\big)$ is a group.*

Proof. Let the assumptions of the theorem hold.
(i) Closure. Let $y, z \in (R^{(b,c)})^{\otimes}$. Then there exist $a, d \in R^{(b,c)}$, such that $y = a^{(b,c)}$ and $z = d^{(b,c)}$. Since $1 \in R^{(b,c)}$, we can apply Theorem 3.1 (b), i.e. we get:

$$y \cdot z = a^{(b,c)}d^{(b,c)} = ((a^{(b,c)})^{(b,c)})^{(b,c)}((d^{(b,c)})^{(b,c)})^{(b,c)}. \tag{11}$$

Now, after applying Corollary 3.1, we get that $(d^{(b,c)})^{(b,c)}(a^{(b,c)})^{(b,c)} \in R^{(b,c)}$ and

$$\Big((d^{(b,c)})^{(b,c)}(a^{(b,c)})^{(b,c)}\Big)^{(b,c)} = ((a^{(b,c)})^{(b,c)})^{(b,c)}((d^{(b,c)})^{(b,c)})^{(b,c)}. \tag{12}$$

Now, by (11) and (12), we get:

$$\Big((d^{(b,c)})^{(b,c)}(a^{(b,c)})^{(b,c)}\Big)^{(b,c)} = y \cdot z.$$

Therefore, $y \cdot z \in (R^{(b,c)})^{\otimes}$.
(ii) Associativity. Let $y, z, w \in (R^{(b,c)})^{\otimes}$. Then there exist $a, d, g \in R^{(b,c)}$, such that $y = a^{(b,c)}$, $z = d^{(b,c)}$ and $w = g^{(b,c)}$. Thereby,

$$(y \cdot z) \cdot w = (a^{(b,c)}d^{(b,c)})g^{(b,c)} = a^{(b,c)}(d^{(b,c)}g^{(b,c)}) = y \cdot (z \cdot w).$$

(iii) Identity. By Lemma 2.7 (c), for arbitrary $a \in R^{(b,c)}$, $1^{(b,c)}a^{(b,c)} = a^{(b,c)}1^{(b,c)} = a^{(b,c)}$. Hence, for every $y \in (R^{(b,c)})^{\otimes}$, we have:

$$1^{(b,c)}y = y1^{(b,c)} = y, \tag{13}$$

i.e. $1^{(b,c)}$ is a unity in $(R^{(b,c)})^{\otimes}$. Let us prove that $1^{(b,c)}$ is the unique unity in $(R^{(b,c)})^{\otimes}$. Suppose that there exists $e \in (R^{(b,c)})^{\otimes}$, such that:

$$y = ey = ye, \quad \text{for every } y \in (R^{(b,c)})^{\otimes}.$$

Hence, the last equality holds for $y = 1^{(b,c)}$. Thus, we get:

$$1^{(b,c)} = e1^{(b,c)} = 1^{(b,c)}e. \tag{14}$$

Since $e \in (R^{(b,c)})^{\otimes}$, the equality (13) is valid for $y = e$. Thus,

$$1^{(b,c)}e = e1^{(b,c)} = e. \tag{15}$$

Now, from (14) and (15), it follows that $e = 1^{(b,c)}$. Hence, $1^{(b,c)}$ is the unique unity in $(R^{(b,c)})^{\otimes}$.
(iv) Inverse. Let $y \in (R^{(b,c)})^{\otimes}$. Then there exists $a \in R^{(b,c)}$, such that $y = a^{(b,c)}$. By Theorem 3.1 (c), we have that:

$$a^{(b,c)}(a^{(b,c)})^{(b,c)} = (a^{(b,c)})^{(b,c)}a^{(b,c)} = 1^{(b,c)}.$$

Hence, we get that $(a^{(b,c)})^{(b,c)}$ is an inverse of $y = a^{(b,c)}$. Moreover, this inverse is unique. Indeed, suppose that:

$$a^{(b,c)}v = va^{(b,c)} = 1^{(b,c)},$$

for some $v \in (R^{(b,c)})^{\otimes}$. Thereby, there exists some $h \in R^{(b,c)}$, such that $v = h^{(b,c)}$. Therefore,

$$a^{(b,c)}h^{(b,c)} = h^{(b,c)}a^{(b,c)} = 1^{(b,c)}.$$

Now, by Theorem 3.10, it follows that $v = h^{(b,c)} = (a^{(b,c)})^{(b,c)}$.

□

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Disclosure Statement

The authors report there are no competing interests to declare.

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