Exact bit error rate of QAM over Nakagami fading channel

Milica I. Petković, Bata V. Vasić, and Milan S. Marković

Abstract — **In this paper, we derive the new closed-form expressions for bit error rate (BER) in detecting quadrature amplitude modulation (QAM) signals transmitted over flat Nakagami-m fading channels. By using this expressions, the effects of fading severity and average signal-to-noise ratio on BER perfomance are analyzed.**

Key word— **Quadrature amplitude modulation, bit error rate, Nakagami fading.**

I. Introduction

FAST development of modern communication techniques causes the demand for reliable high date rate transmission, which stimulate much interest in modulation techniques. Quadrature amplitude modulation (QAM) is one of widely used modulation techniques because of its efficiency in power and bandwidth [1]-[3]. The analyisis of the bit error rate (BER) performance of a one-dimensional pulse amplitude modulation (PAM) over aditive white Gaussian noise (AWGN) channel was presented in [4]. Also, this paper presented the results about the BER performance analysis of a two-dimensional amplitude modulation, M-ary square QAM and an IxJ rectangular QAM signals over AWGN channel.

In wireless communication, the variation of instantaneous value of the received signal, i.e. fading is one of the main causes of performance degradation. Recently, reception through a channel under the influence of Nakagami fading occupied the attention of many researches for several reasons. Nakagami distribution has a relatively simple analytical form, making it convinient in performance analysis. It can be used to account for both severe and weak fading and includes the classical Rayleigh fading as a special case. Several works have been reported on the performance analysis of QAM in Nakagami fading channels, where mainly the symbol error rate (SER) performance has been derived [3], [5], [6].

To the best of the authors' knowladge, the exact analytical closed-form expressions for the bit error rate (BER) have not been presented in the technical literature up to now. In this paper, the new closed-form expressions for BER in detecting QAM signals transmitted over Nakagami fading channel are derived. The BER dependence on fading severity and average signal-to-noise ratio (SNR) per bit is presented.

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II. System model

The M-ary square QAM signal is a two-dimensional generalization of a one-dimensional M-ary PAM signal. The QAM signal consists of two independently amplitude-modulated carriers in quadrature expressed by [1], [3]

$$s(t) = A_I \cos 2\pi f_c t - A_j \sin 2\pi f_c t,\ 0 \le t \le T\ , \quad (1)$$

where A_I and A_J are the amplitudes of in-phase and quadrature components, f_c is the carrier frequency, and T is the symbol period. Depending on the number of possible symbols M, two distinct QAM constellations can be distinguish: square constellations with even number of bits per symbol, and rectangular constellations where the number of bits per symbol is odd.

In M-ary square QAM, $\log_2 M$ bits of the serial information stream are mapped on a two-dimensional signal constellation using Gray coding. In (1), A_I and A_J are selected independently over the set $\left\{\pm d, \pm 3d, ..., \pm(\sqrt{M}-1)d\right\}$ where $2d$ is the Euclidean distance between two adjacent signal points and is given by

$$d = \sqrt{\frac{3\log_2 M \cdot E_b}{2(M-1)}} \quad (2)$$

where E_b is the bit energy.

The arbitrary $I \times J$ rectangular QAM format consists of two PAM schemes, I-ary and J-ary PAM. The relation between the Euclidean distance of the nearest signal points is now

$$d = \sqrt{\frac{3\log_2 (I \cdot J) \cdot E_b}{I^2 + J^2 - 2}} \quad (3)$$

Signal is transmitted from the transmitter to the receiver via channel with Nakagami-m fading. The received signal envelope can be described by Nakagami distribution given by [3]

$$p_r(r) = \frac{2m^m r^{2m-1} e^{-\frac{m}{\Omega}r^2}}{\Gamma(m)\Omega^m}, \quad x > 0, \quad (4)$$

where m is the Nakagami parameter and Ω is the average power $\Omega = \mathrm{E}\{r^2\}$ with E denoting mathematical expetaction. On the basis of (4) one can show that instantenous SNR has the gamma distribution given by

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$$p_\gamma(\gamma)=\frac{m^m\gamma^{m-1}e^{-\frac{m}{\Omega}\gamma}}{\Gamma(m)\Omega^m},\quad \gamma>0. \tag{5}$$

where $\Omega = \gamma_0 = E\{\gamma\}$ is the average SNR.

III. BER ANALYSIS OF QAM OVER THE NAKAGAMI FADING CHANNEL

Based on (1), signal on the reception of the channel with the Nakagami fading is

$$s_{r(t)} = r\cdot A_I\cos 2\pi f_c t - r\cdot A_j \sin 2\pi f_c t,\ 0\le t\le T, \tag{6}$$

where r is the envelope of received signal that has the Nakagami distribution given by (4).

A. General BER expression of M-ary square QAM

Using [4, eq. (14)], the conditional kth bit error probability of M-ary square QAM can be expressed by

$$P_{b/\gamma}(k)=\frac{1}{\sqrt{M}}\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1}\left\{(-1)^{\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}\right\rfloor}\times\left(2^{k-1}-\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}+\frac{1}{2}\right\rfloor\right)\times\operatorname{erfc}\left((2i+1)\sqrt{\frac{3\log_2 M\cdot\gamma}{2(M-1)}}\right)\right\} \tag{7}$$

where erfc(.) is the complementary error function and γ denotes the instantenous SNR per bit. Since the instantenous SNR per bit is random variable, in order to obtain average kth bit error probability of M-ary square QAM, it is necessary to average previous expression. So, we have

$$P_b(k)=\int_{\gamma=0}^{\infty}P_{b/\gamma}(k)p_\gamma(\gamma)d\gamma \tag{8}$$

Now, using (7) and (8), the bit error probability $P_b(k)$ (kth bit is in error) can be expressed as

$$P_b(k)=\frac{1}{\sqrt{M}}\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1}\left\{(-1)^{\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}\right\rfloor}\times\left(2^{k-1}-\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}+\frac{1}{2}\right\rfloor\right)\times\int_{\gamma=0}^{\infty}\operatorname{erfc}\left((2i+1)\sqrt{\frac{3\log_2(M)\gamma}{2(M-1)}}\right)p_\gamma(\gamma)d\gamma\right\} \tag{9}$$

where γ is the instantenous SNR per bit and $p_\gamma(\gamma)$ is given by (5).

Integral in (9) can be solved by representing complementary error function and exponential function in terms of Meijer's G functions by using [7, eqs. 01.03.6.0004.01 and 06.27.26.0006.01], and after that by using [7, 07.34.21.0011.01]. The eq. (9) becomes

$$P_b(k)=\frac{1}{\sqrt{M}}\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1}\left\{(-1)^{\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}\right\rfloor}\times\left(2^{k-1}-\left\lfloor\frac{i\cdot 2^{k-1}}{\sqrt{M}}+\frac{1}{2}\right\rfloor\right)\times\frac{\left(\frac{2m(M-1)}{(2i+1)^2\cdot 3\log_2 M\cdot\gamma_0}\right)^m}{\Gamma(m)\cdot\Omega^m\cdot\sqrt{\pi}}\times G_{2,2}^{1,2}\left(\frac{2(M-1)m}{(2i+1)^2 3\log_2(M)\gamma_0}\,\middle|\,\begin{matrix}1-m, & \frac{1}{2}-m\\ 0, & -m\end{matrix}\right)\right\} \tag{10}$$

Using [4, eq. (16)], the extact expresion of average BER of M-ary square QAM is given by

$$P_b=\frac{1}{\log_2\sqrt{M}}\sum_{k=1}^{\log_2\sqrt{M}}P_b(k) \tag{11}$$

where $P_b(k)$ is given by (10).

B. General BER expression of M-ary rectangular QAM

In the similar way, using [4, eq (20), (21), (22)], the average BER of M-ary rectingular QAM can be determined. The average BER of $I\times J$ rectangular QAM over Nakagami fading channel is

$$P_b=\frac{1}{\log_2(I\cdot J)}\left(\sum_{k=1}^{\log_2 I}P_I(k)+\sum_{l=1}^{\log_2 J}P_J(l)\right) \tag{12}$$

where

$$P_I(k)=\frac{1}{I}\sum_{i=0}^{(1-2^{-k})I-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{I}\right\rfloor}\times\left(2^{k-1}-\left\lfloor\frac{i2^{k-1}}{I}+\frac{1}{2}\right\rfloor\right)\times\frac{\left(\frac{m(I^2+J^2-2)}{(2i+1)^2 3\log_2(IJ)\gamma_0}\right)^m}{\Gamma(m){\gamma_0}^m\sqrt{\pi}}\times G_{2,2}^{1,2}\left(\frac{m(I^2+J^2-2)}{(2i+1)^2 3\log_2(IJ)\gamma_0}\,\middle|\,\begin{matrix}1-m, & \frac{1}{2}-m\\ 0, & -m\end{matrix}\right)\right\} \tag{13}$$

and

$$
\begin{aligned}
P_J(l) = \frac{1}{J}\sum_{j=0}^{(1-2^{-l})J-1} \Bigg\{ & (-1)^{\left\lfloor \frac{j2^{l-1}}{J} \right\rfloor} \\
& \times \left(2^{l-1} - \left\lfloor \frac{j2^{l-1}}{J} + \frac{1}{2} \right\rfloor\right) \\
& \times \frac{\left(\dfrac{m(I^2+J^2-2)}{(2i+1)^2 3\log_2(IJ)\gamma_0}\right)^m}{\Gamma(m)\gamma_0{}^m\sqrt{\pi}} \\
& \times G_{2,2}^{1,2}\left(\frac{m(I^2+J^2-2)}{(2i+1)^2 3\log_2(IJ)\gamma_0}\,\middle|\, \begin{matrix} 1-m, & \frac{1}{2}-m \\ 0, & -m \end{matrix}\right)\Bigg\} \qquad (14)
\end{aligned}
$$

IV. NUMERICAL RESULTS

In this section it will be consider how BER depends on average SNR per bit in channel with the Nakagami fading during *M*-ary QAM signal transmission. It will be also observed the effect of the Nakagami parameter *m* on BER.

The numerical results are obtained by expressions (11) and (10) for M-ary square QAM, and (12), (13) and (14) for M-ary rectangular QAM.

Fig. 1 shows BER dependence on average SNR in the Nakagami channel with different values of the Nakagami fading parameter during 4QAM transmission. For higher values of SNR, the BER is lower. The performance of the system is the worst for *m*=0.5. When the value of the Nakagami parameter *m* is lower, the depth of fading is bigger.

Fig. 2 shows the same dependence for rectangular 8QAM. The conclusion is the same as for 4QAM. Figures 3 and 4 show this dependence for 16QAM and 32QAM, respectively. With decreasing the value of the Nakagami parameter *m*, we have severe fading.

Fig. 5 shows the BER dependence on average SNR in the Nakagami fading, *m*=1, for square 4QAM, 16QAM and 64QAM and rectangular 8QAM,32QAM and 128 QAM.

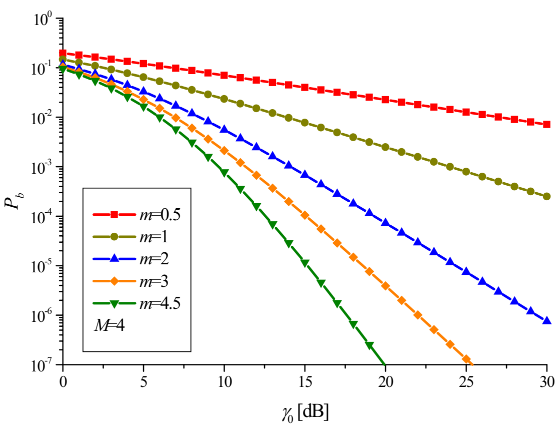

Fig. 1. BER dependence on average SNR in the Nakagami fading channel with different values of the Nakagami fading parameter

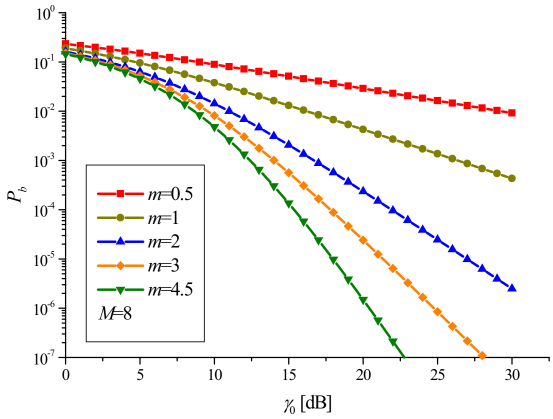

Fig. 2. BER dependence on average SNR in the Nakagami fading channel with different values of the Nakagami fading parameter during 8QAM

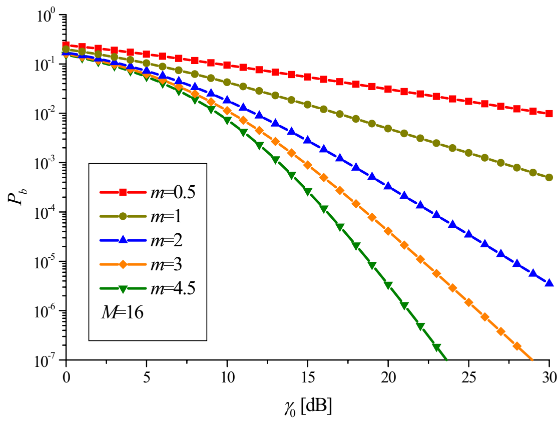

Fig. 3. BER dependence on average SNR in the Nakagami fading channel with different values of the Nakagami fading parameter during 16QAM

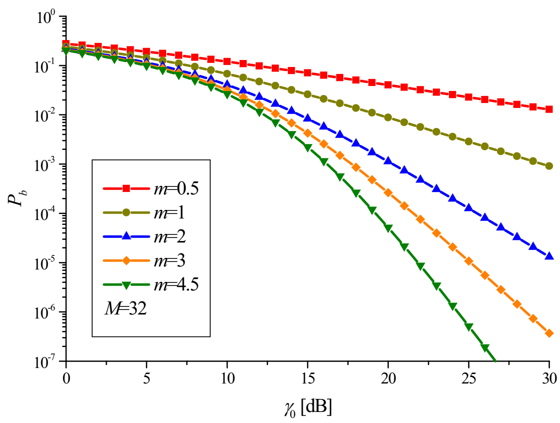

Fig. 4. BER dependence on average SNR in the Nakagami fading channel with different values of the Nakagami fading parameter during 32QAM

The same dependence, but for the Nakagami parameter *m*=3 is shown in Fig.6. When the average SNR is 20 dB, the values of BER for 4QAM, 8QAM and 32QAM are 3.95×10^{-6}, 2.4×10^{-5} and 2.710^{-4}, respectively. With higher order of QAM, the performance of BER is worse, but the larger amount of information is transmitted.

As Fig. 5 and 6 show, with deeper fading (*m*=1) the BER curves of different QAM order are closer than for case of *m*=3. When the impact of fading is lower, the distance between the BER curves is bigger.

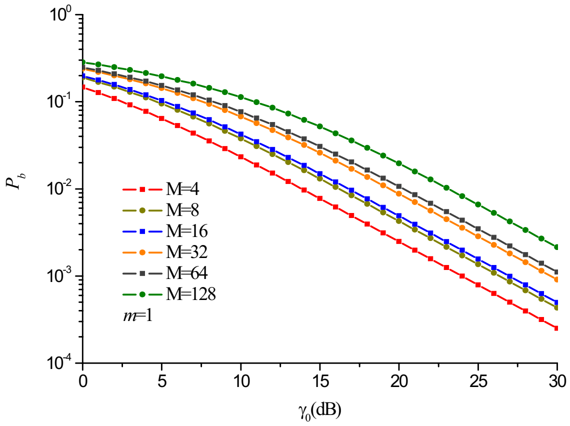

Fig. 5. BER dependence on average SNR in the Nakagami fading channel with the Nakagami fading parameter *m*=1 for different types of QAM (*M*=4,8,16,32,64,128)

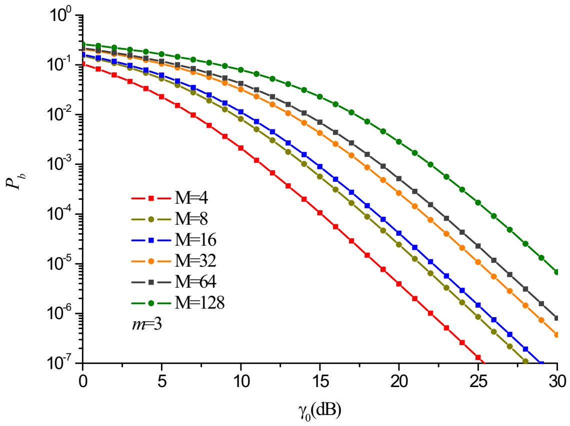

Fig. 6. BER dependence on average SNR in the Nakagami fading channel with the Nakagami fading parameter m=3 for different types of QAM (M=4,8,16,32,64,128)

V. Conclusion

In this paper we have analyzed *M*-ary QAM transmisssion over the channel with the Nakagami fading. The closed-form exprresions for BER have been derived and used for observing the BER performances. The effects of the Nakagami parameter and the order of QAM modulation on the BER have been noted.

Acknowledgement

This paper was supported in part by the Ministry of Science of Republic of Serbia under grant TR-32028.

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