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On the similarity of some three-point methods for solving nonlinear equations

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ABSTRACT

In this short note we discuss certain similarities between some three-point methods for solving nonlinear equations. In particular, we show that the recent three-point method published in [R. Thukral, A new eighth-order iterative method for solving nonlinear equations, Appl. Math. Comput. 217 (2010) 222–229] is a special case of the family of three-point methods proposed previously in [R. Thukral, M.S. Petković, Family of three-point methods of optimal order for solving nonlinear equations, J. Comput. Appl. Math. 233 (2010) 2278–2284].

In this short note we point to certain similarities between recent three-point methods of the optimal order eight. We show that the three-point method recently published in [1] is a special case of the family of three-point methods proposed in [2]. We also give a comment on the similarity of the methods presented in [7] with the methods previously stated in the already mentioned paper [2].

Multipoint iterative methods for solving nonlinear equations $f(x) = 0$ were extensively studied in Traub's book [3] and some papers published in the 1960s and 1970s. The interest for multipoint methods has renewed in the first decade of the 21st century because this class of methods overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency.

During the last two years several optimal three-point methods using only four function evaluations have been developed, see [2,4–10]. One of them is the family of eight-order methods proposed in [2] in the form:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)+bf(y_k)}{f(x_k)+(b-2)f(y_k)}, \qquad (a, b \in \boldsymbol{R}, k = 0, 1, \ldots) \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}\left[\varphi\left(\frac{f(y_k)}{f(x_k)}\right) + \frac{f(z_k)}{f(y_k)-af(z_k)} + \omega\left(\frac{f(z_k)}{f(x_k)}\right)\right], \end{cases} \tag{1}$$

where φ and ω are arbitrary real functions satisfying the conditions

$$\varphi(0) = 1, \quad \varphi'(0) = 2, \quad \varphi''(0) = 10 - 4b, \quad \varphi'''(0) = 12b^2 - 72b + 72, \quad \omega(0) = 0, \quad \omega'(0) = 4 \tag{2}$$

and a and b are real parameters. Let us note that the first two steps form the King family of fourth-order methods [11]. The presented family (1) gives a variety of three-point methods since, apart from the parameters a and b, the parametric functions φ and ω satisfying the condition (2) can be chosen in different ways.

Very recently, the first author of the family (1) published in [1] the three-point methods named as "the new Newton-type method (N^m)" taking $m = 5, 6, 7$ and 8. All of these methods require four function evaluations per iteration and have the order m. Obviously, according to the Kung–Traub hypothesis [12], the methods of the type (N^5), (N^6), and (N^7) are not optimal and

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they are not of interest from a practical point of view. For this reason, we are concentrating on the only optimal method, referred to as "the new Newton-type eight-order method (N^8)", which has the form

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \frac{f(x_k)^2+f(y_k)^2}{f'(x_k)(f(x_k)-f(y_k))}, & (k=0,1,\ldots) \\ x_{k+1} = z_k - \left[\left(\frac{1+\mu_k^2}{1-\mu_k}\right)^2 - 2\mu_k^2 - 6\mu_k^3 + \frac{f(z_k)}{f(y_k)} + 4\frac{f(z_k)}{f(x_k)}\right]\frac{f(z_k)}{f'(x_k)}, \end{cases} \tag{3}$$

where $\mu_k = f(y_k)/f(x_k)$.

It is easy to show that the choice $b = 1$ in the second step of (1) gives

$$z_k = x_k - \frac{f(x_k)^2 + f(y_k)^2}{f'(x_k)(f(x_k) - f(y_k))},$$

which is equivalent to the second step in (3). Note that the first two steps of (3) define the two-point method rediscovered by Kou et al. [13].

Let the functions φ and ω appearing in (1) be of the form

$$\varphi(t) = \left(\frac{1+t^2}{1-t}\right)^2 - 2t^2 - 6t^3, \quad \omega(t') = 4t', \tag{4}$$

where $t = f(y)/f(x)$ and $t' = f(z)/f(x)$. From (4) we find

$$\varphi(0) = 1, \quad \varphi'(0) = 2, \quad \varphi''(0) = 6, \quad \varphi'''(0) = 12, \quad \omega(0) = 0, \quad \omega'(0) = 4,$$

which coincides with the conditions (2) for $b = 1$, already chosen in the second step of (1). Let us note that the function φ given by (4) was installed in (3) without clear motivation, derivation and citation of an origin (see [1]).

According to the previous consideration, we can conclude that the method (3) is a special case of the family (1) for $b = 1$, and φ and ω given by (4). Formally, we can take $a = 0$ in (1) but this choice is irrelevant.

Speaking about the generalized method (1) and its special cases, we may consider another three-point method with similar structure published in [7]

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(y_k)}{f'(x_k)} \cdot \frac{f(x_k)}{f(x_k)-2f(y_k)}, & (a \in \boldsymbol{R}, k=0,1,\ldots) \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}\left[\left(\frac{f(x_k)-f(y_k)}{f(x_k)-2f(y_k)}\right)^2 + \frac{f(z_k)}{f(y_k)-af(z_k)} + G(t_k)\right], \end{cases} \tag{5}$$

where $t_k = f(z_k)/f(x_k)$ and G is an arbitrary real function satisfying the conditions $G(0) = 0$, $G'(0) = 4$. The method (5) and a variant of (5) with a special choice $G(t) = 4t/(1 + \alpha_2 t)$ ($\alpha_2 \in \boldsymbol{R}$) were referred to as new methods in [7].

Observe that the second step of (5) is obtained from the second step of (1) setting $b = 0$ (giving Ostrowski's method [14]). Besides, the parametric function φ in the third step of (1) gives a number of different methods. Moreover, having in mind that $b = 0$ and taking the function

$$\varphi(t) = \left(\frac{1-t}{1-2t}\right)^2 \quad (t = f(y)/f(x)) \tag{6}$$

in (1), we observe that the conditions (2) are satisfied (for $b = 0$). But the choice of φ defined by (6) gives the first term of the expression within the square brackets in the third step of (5). Since the functions G (in (5)) and ω (in (1)) satisfy the same conditions, it is obvious that the three-point method (5) is a special case of (1).

The method (5) was generalized in the paper [7] in the form

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(y_k)}{f'(x_k)} H\left(\frac{f(y_k)}{f(x_k)}\right), \\ x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}\left[U\left(\frac{f(y_k)}{f(x_k)}\right) + V\left(\frac{f(z_k)}{f(y_k)}\right) + W\left(\frac{f(z_k)}{f(x_k)}\right)\right], \end{cases} \tag{7}$$

where H, U, V and W are real-valued functions. It is easy to note that the iterative method (7) essentially combines Chun's two-step method [15]

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \frac{f(y_k)}{f'(x_k)}\left[h\left(\frac{f(y_k)}{f(x_k)}\right)\right]^{-1} \end{cases} \tag{8}$$

and the third step of the method (1).

Finally, let us note that the paper [2] was not cited in the papers [1] and [7]. Chun's method (8) was formally included in the list of references in [7], but it was not quoted anywhere in the paper.

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